

Exam QFIQF

Date: Thursday, October 24, 2024

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 12 questions numbered 1 through 12 with a total of 70 points.

The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.
3. Each question part or subpart should be answered either in the Excel document or the paper provided as directed. Graders will only look at the work as indicated.
4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).
5. Prior to uploading your Word and Excel files, each file should be saved and renamed with your unique candidate number in the filename. To maintain anonymity, please refrain from using your name and instead use your candidate number.
6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

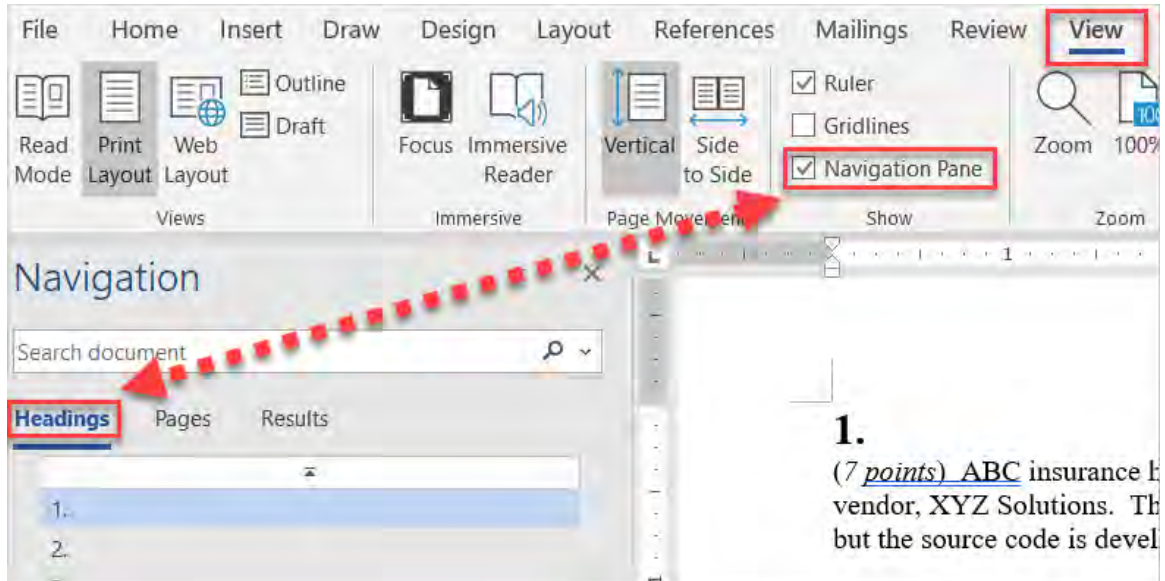
Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.
2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.
3. The answer should be confined to the question as set.
4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.
5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(6 points) Let $\{B_t: 0 \leq t \leq T\}$ be a Wiener process with respect to the filtration $\{I_t: 0 \leq t \leq T\}$, where $T > 0$ is some future date.

Let $\{X_t: 0 \leq t \leq T\}$ and $\{Y_t: 0 \leq t \leq T\}$ be stochastic processes defined as follows:

$$X_t = \int_0^t 1_{\{B_u > 0\}} dB_u$$
$$Y_t = \int_0^t 1_{\{B_u < 0\}} dB_u$$

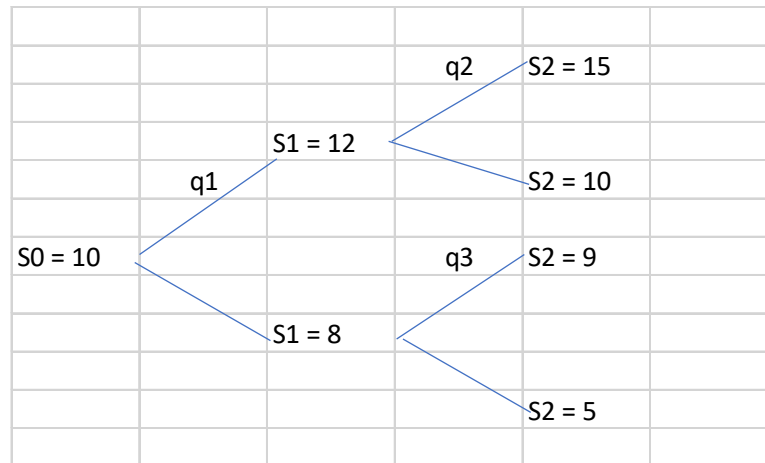
where 1_A is the indicator function for event A .

- (a) (1 point) Calculate $E[X_t^2]$ for $t < T$.
- (b) (0.5 points) Calculate $E[X_t Y_t]$ for $t < T$.
- (c) (3 points)
 - (i) (0.5 points) List the three properties of a martingale.
 - (ii) (2.5 points) Determine whether $\{X_t Y_t: 0 \leq t \leq T\}$ is a martingale with respect to the filtration $\{I_t: 0 \leq t \leq T\}$ by verifying whether all the three properties listed in part (c)(i) hold.
- (d) (1.5 points) Show, using Jensen's Inequality, that $E[\ln(|X_t B_t|)] \leq \ln(t)$ for $0 < t \leq T$.

The responses for all parts of this question are required on the paper provided to you.

2.

(5 points)



Consider a two-period discrete time model where the interest rate for a period is r and the price of the risky asset evolves according to the above diagram.

(a) (2.5 points)

- (i) Determine the range of r for which this model is arbitrage-free.
- (ii) Assess whether this model is complete for the range of r in part (a)(i).

Consider a look-back call option which has the following payoff:

$$C = (\max(S_1, S_2) - K)^+$$

The strike price of the option is $K = 11$.

- (b) (2.5 points) Calculate the fair price of this option when $r = 1/9$ using the risk-neutral measure.

The responses for all parts of this question are required on the paper provided to you.

3.

(4 points) Let $\{N_t: t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration \mathcal{F}_t .

The compensated Poisson process is defined by:

$$\widehat{N}_t = N_t - \lambda t.$$

- (a) (1 point) Show that \widehat{N}_t is a martingale.
- (b) (2 points) Show that $\widehat{N}_t^2 - \lambda t$ is a martingale.
- (c) (1 point) Determine whether $X_t = \exp(\widehat{N}_t)$ is a martingale.

The responses for all parts of this question are required on the paper provided to you.

4.

(7 points) The level $S(t)$ of an equity index IND at time t follows a geometric Brownian motion with $S(0) = 1$:

$$dS(t) = r S(t)dt + \sigma S(t)dW(t)$$

where r is the constant risk-free rate, σ the constant volatility, and $W(t)$ a standard Brownian motion under the risk-neutral measure.

(a) (1 point) Derive the SDE for $S(t)^\alpha$, where α is a constant and $0 < \alpha < 1$.

(b) (1 point) Show $\Pr(S(t)^\alpha \leq e^{gt}) = \Phi\left(\frac{-(r-\frac{1}{2}\sigma^2)t+\frac{gt}{\alpha}}{\sigma\sqrt{t}}\right)$, where g is a constant.

Your company is reviewing different crediting strategies for a single premium equity indexed annuity with IND as the reference index, and the minimum guaranteed rate $g=0\%$. Policyholders could choose either of following strategies:

Crediting strategy 1 is Point-to-Point Option design:

Payoff at Maturity $T = \max(S(T)^\alpha, 1)$ where α is the participation rate.

Crediting strategy 2 is a Double Threshold design:

The payoff at maturity T is:

$$\begin{aligned} S(T)^{\alpha_2} & \quad \text{if } S(T) \geq e^{B_2 T/\alpha_2} \\ S(T)^{\alpha_1} & \quad \text{if } e^{B_1 T/\alpha_1} \leq S(T) < e^{B_2 T/\alpha_2} \\ 1 & \quad \text{if } S(T) < e^{B_1 T/\alpha_1} \end{aligned}$$

where $B_1, B_2, \alpha_1, \alpha_2$ are positive constants less than 1.

$$B_2 \geq B_1, \quad \frac{B_2}{\alpha_2} > \frac{B_1}{\alpha_1}$$

4. Continued

(c) (1 point) Show that the price P of the contract, under the classic point-to-point design, is equal to $e^{-rT} \left[\Pr(\ln S(T) \leq 0) + \mathbb{E}[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}] \right]$

(d) (1 point) Prove that

$$\mathbb{E}[S(T)^\alpha \mathbb{I}_{\{\ln S(T) > 0\}}] = e^{\alpha(r - \frac{1}{2}\sigma^2)T} \int_{-\frac{(r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}^{\infty} e^{\alpha\sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

You are given as a hint that for fixed values A and γ ,

$$\int_A^{\infty} e^{\gamma z} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right) dz = e^{\frac{\gamma^2}{2}} \Phi(\gamma - A)$$

where Φ denotes the cumulative function of the standard normal distribution.

(e) (0.5 points) Show that

$$P = e^{-rT} \left[\Phi \left(\frac{-\left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) + e^{\alpha\left(r + \frac{1}{2}(\alpha-1)\sigma^2\right)T} \Phi \left(\frac{\left(r - \frac{1}{2}\sigma^2 + \alpha\sigma^2\right)T}{\sigma\sqrt{T}} \right) \right]$$

Your marketing colleagues believe the classic point-to-point design has low demand in the current market. They suggest a double-barrier design as an alternative to better entice prospective customers. Given recent market volatility, they also recommend we target 1-year maturities.

(f) (2.5 points) Derive the arbitrage-free price of the double threshold design for $T = 1$, using the results from part (e).

The responses for some parts of this question are required on the paper provided to you.

5.

(7 points) You are a manager of a fixed income portfolio and asked to dynamically replicate the portfolio.

- (a) (1 point) Describe the application of the replicating portfolio in the fixed-income context for the perspective of risk management of derivative hedge and the relative value trading on the yield curve.
- (b) (1 point) Describe the difference between the risk management for derivative hedge and relative value trading cases for the following aspects:
 - (i) The set up of a replicating portfolio.
 - (ii) The underlying asset positions in the replicating portfolios as the derivative in hedge and original bond in relative trading approach their maturities.

Consider two zero-coupon bonds (Bond A and Bond B) that the underlying interest rate follows the Vasicek interest rate model.

$$dr_t = \gamma^*(\bar{r}^* - r_t)dt + \sigma dX_t$$

The following values apply:

- Initial interest rate level (r_0): 5.00%
- Long-term mean interest rate (\bar{r}^*): 4.00%
- Speed of mean reversion (γ^*): 0.4653
- Volatility (σ): 3.2%
- T_A (Maturity of Bond A): 1.5 years
- T_B (Maturity of Bond B): 2.5 years

5. Continued

You have been tasked with creating a replicating portfolio for Bond A using Bond B and cash. The time step is one month, and you need to calculate the replicating portfolio position for a year. The Vasicek bond price formula is given:

$$Z(r, t; T) = e^{A(t;T) - B(t;T)r}$$

$$B(t; T) = \frac{1}{\gamma^*} (1 - e^{-\gamma^*(T-t)})$$

$$A(t; T) = (B(t; T) - (T - t)) \left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2} \right) - \frac{\sigma^2 (B(t; T))^2}{4\gamma^*}$$

Assume interest rate for each monthly time step follows deterministic as $r_{t+1} = r_t + \gamma^*(\bar{r}^* - r_t) * \Delta t$

(c) (4 points)

- (i) (1.5 points) Calculate the interest rate, Bond A price, Bond B price, the optimal hedge ratio, cash needed for rebalancing and interest received for each month.

The response for this part is to be provided in the Excel spreadsheet.

- (ii) (2 points) Tabulate the results in a detailed table format using the provided calculated table.

The response for this part is to be provided in the Excel spreadsheet.

- (iii) (0.5 points) Plot a chart displaying the hedge ratio and the cash position.

The response for this part is to be provided in the Excel spreadsheet.

Assume the portfolio is already delta hedged.

(d) (1 point)

- (i) Explain the theta-gamma relationship.
- (ii) Explain its implications for dynamic hedging.

The responses for all parts of this question are required on the paper provided to you.

6.

(6 points) You are given the following Hull-White model, which is calibrated to the current market bond prices, to value at time 0, 2.1-year European call option on a semi-annual coupon bond that will mature in 3 years.

$$dr_t = (\theta_t - \gamma^* r_t)dt + \sigma dX_t,$$

where γ^* and σ are positive constants, and X_t is a standard Brownian motion.

The price at time t of a zero-coupon bond with \$1 principal with maturity date $T \geq t$ is given by $Z(r, t; T) = e^{A(t; T) - B(t; T)r}$,

$$\text{where } B(t; T) = \frac{1 - e^{-\gamma^*(T-t)}}{\gamma^*} \text{ and } A(t; T) = \frac{a}{\gamma^*} (1 - e^{-\gamma^*(T-t)}) - a(T-t) - \frac{b(T-t)^2}{2}.$$

You are given the following:

- $\sigma = 0.2$, $\gamma^* = 0.309105$
- $a = 0.04$, $b = 0.0862$
- $r_0 = \text{Initial short-term interest rate} = 4\%$

(a) (1 point) Calculate

- (i) $B(2.1; 2.5)$ and $B(2.1; 3)$
- (ii) $A(2.1; 2.5)$ and $A(2.1; 3)$

Let $P(r, t)$ be the price at time $t \leq 3$ of a semi-annual coupon bond with

- Annual coupon rate = 2%
- Principal = \$100
- Maturity $T = 3$

You are given $r_K^* = 4.08541\%$.

(b) (2 points)

- (i) Calculate $Z(r_K^*, 2.1; 2.5)$
- (ii) Calculate $Z(r_K^*, 2.1; 3)$
- (iii) Show that $P(r_K^*, 2.1) = 95$

6. Continued

Consider a European Call option on the coupon bond with strike 95 to be matured in 2.1 years.

You are given the following:

- The price at $t=0$ of the 2.1-year European call with strike price of $Z(r_K^*, 2.1; 3)$, written on zero coupon bond with face value \$1 and maturity of 3 years is 0.010116858.
- (c) (3 points) Compute the value at $t=0$ of the above European Call option on the coupon bond.

The responses for all parts of this question are required on the paper provided to you.

7.

(7 points) You are given one-month daily treasury bill yields (annualized) over 500 consecutive trading days in the `daily_data` table. There are 252 trading days per year. You would like to fit the CIR model,

$$dr = \gamma(\bar{r} - r)dt + \sqrt{\alpha r} dX$$

for the data set.

For this model you are considering the method based on Euler discretization and the method based on the transition density function.

(a) (3 points) Compare and contrast these two methods.

You are given the attached R code snippets with outputs.

(b) (2 points) Calculate the estimates of γ , \bar{r} , and α based on Euler discretization.

(c) (1 point) Write estimates of γ , \bar{r} , and α based on the transition density method.

(d) (1 point) Recommend an estimate method between Euler discretization method and the transition density method.

Partially completed R codes and outputs.

```
guess = c(4,0.1,0.1)
model1 = optim(c(gamma=4,rbar=0.1,alpha=0.1),
              CIR.log.lik,NULL,
              daily_data$Yield,method="BFGS")
model1$par
##      gamma      rbar      alpha
## 7.8697698 0.0533068 0.2674647
options()$nwarnings
## [1] 50
model1$count
## function gradient
##      83      18
```

7. Continued

Partially completed R codes and outputs.

```
# use the data frame
# display header and first three rows
head(daily_data,3)

##      Yield
## 1 0.0200000
## 2 0.0186188
## 3 0.0181607

# display header and last two rows
tail(daily_data,2)

##      Yield
## 499 0.1018005
## 500 0.1038554

N= nrow(daily_data)

euler.est= matrix(NA,3,1)
x1 = daily_data$Yield[1:(N-1)]^(-0.5)
x2 = daily_data$Yield[1:(N-1)]^(0.5)
y = daily_data$Yield[2:N]*x1
model=lm(y~0+x1+x2)
summary(model)

##
## Call:
## lm(formula = y ~ 0 + x1 + x2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.037402 -0.010118 -0.000279  0.009438  0.040784
##
## Coefficients:
##      Estimate Std. Error t value Pr(>|t|)
## x1 0.0003346  0.0002136   1.567   0.118
## x2 0.9968652  0.0049121 202.941 <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01455 on 497 degrees of freedom
## Multiple R-squared:  0.9961, Adjusted R-squared:  0.9961
## F-statistic: 6.304e+04 on 2 and 497 DF,  p-value: < 2.2e-16
```

The responses for all parts of this question are required on the paper provided to you.

8.

(5 points) Assume that the dynamics of a stock S_t is described the following SDE:

$$dS_t = rS_t dt + \sigma_R S_t dW_t$$

where r is the constant risk-free rate, σ_R is the realized volatility of S_t , and W_t is a Brownian motion.

You bought a call option with strike K and maturity at time T on S_t at the value C_t , delta-hedged the position based on the implied volatility Σ . Assume the cost of borrowing is the risk-free rate r .

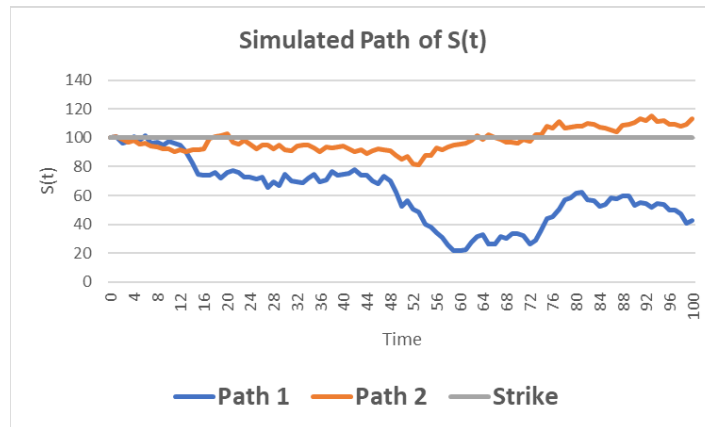
- (a) (1.5 points) Calculate the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt .

Assume that the hypothetical value of the call option is C_t^R if it was valued using the realized volatility, and you have delta-hedged the long position in C_t based on the realized volatility σ_R .

- (b) (2 points)
- (i) Prove that the gain or loss of the hedged portfolio dV_t over an infinitesimal period dt is $dV_t = e^{rt} d[e^{-rt}(C_t - C_t^R)]$
 - (ii) Derive the present value of the total gain or loss to maturity $\int_t^T e^{r(s-t)} dV_s$.

8. Continued

Given two simulated paths of S_t from the given SDE:

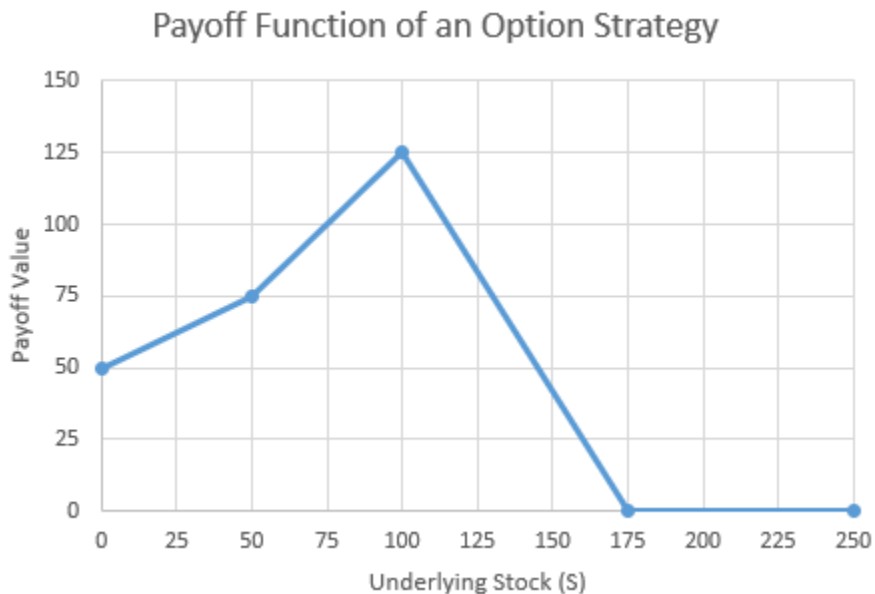


- (c) (1.5 points) Compare $\int_0^{100} e^{r(s-t)} dV_s$ between the two paths if they materialize respectively, assuming
- The portfolio is delta-hedged based on σ_R .
 - The portfolio is delta-hedged based on Σ .

The responses for all parts of this question are required on the paper provided to you.

9.

(4 points) You are given the payoff function of an option strategy at expiration in one year:



Assume the following:

- The risk-free rate is constant and continuously compounded at 5%.
- The spot price of the underlying non-dividend-paying stock is 75.
- The table provides the 1-year call prices at various strike prices determined by Black-Scholes:

Strike (K)	Call Price
0	75.00
50	27.93
100	3.07
175	0.04

- (a) (2 points) Determine the value of this option strategy.
- (b) (1 point) Construct a replicating portfolio that fully hedges the payoff at expiration.

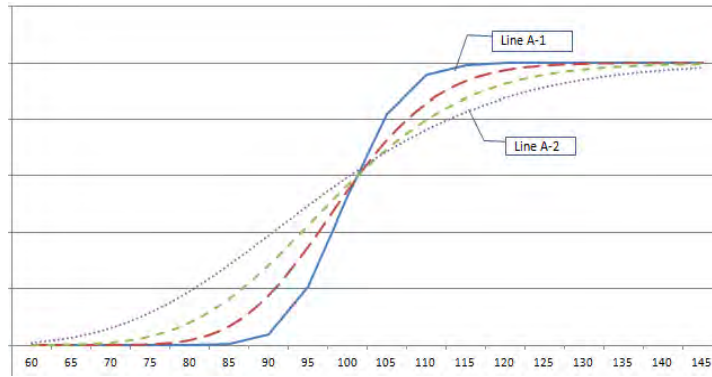
When creating a replicating portfolio, the market price of the call options is more expensive than what are determined using Black-Scholes formula.

- (c) (1 point) Provide two concrete explanations for why this happens in the market.

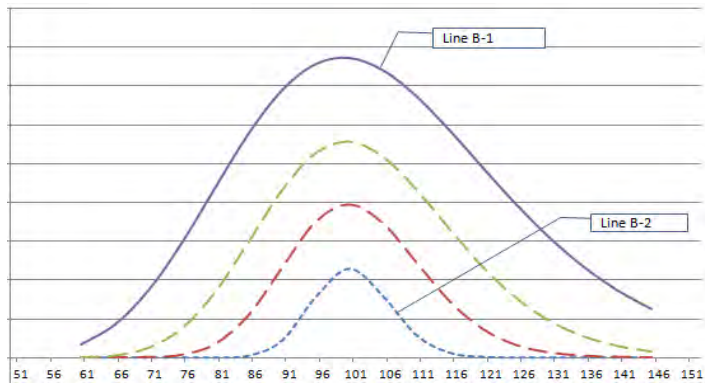
The responses for some parts of this question are required on the paper provided to you.

10.

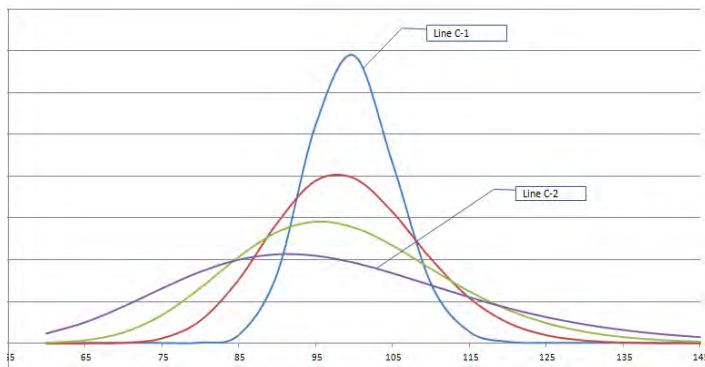
(5 points) Graph A, Graph B, and Graph C show the Greeks of a European put option on a non-dividend-paying stock at different time to expiry based on the Black-Scholes-Merton model.



Graph A



Graph B



Graph C

The above three graphs show the sensitivity of the Greeks with respect to time to expiry.

10. Continued

- (a) (1.5 points) Determine which Greek (Delta, Gamma, or Vega) and which expiry (1-year or 1-month) by filling the table below.

	Greek	Justification to Greek	Expiry (choose 1-year vs. 1-month)	Justification to Expiry
Graph A			Line A-1:	
			Line A-2:	
Graph B			Line B-1:	
			Line B-2:	
Graph C			Line C-1:	
			Line C-2:	

You are given an at-the-money European put option based on a non-dividend-paying stock S with the following parameters:

- Annual continuous risk-free interest rate = 5%
- Time to maturity = 1 year
- Volatility = 20%
- Gamma of the option = 0.03
- Rho of the put option = -3

- (b) (2 points) Determine, using the Black-Scholes-Merton model:
- the Theta if the spot price stayed the same
 - the Vega of the option if the stock price instantly changed to 10%

You are given the following assumptions:

- Current index value $S = 100$
- Annual continuous risk-free rate $r = 2\%$
- Strike level is 80, 100, 120 assuming 100 is the mid-point
- Time to maturity = 1 year

- (c) (1.5 points)
- Describe how to construct a butterfly spread with the strike prices 80, 100, and 120.

The response for this part is to be provided in the Excel spreadsheet.

10. Continued

- (ii) Plot the Vega of the butterfly spread in part (c)(i) as a function of volatility of the underlying stock

The response for this part is to be provided in the Excel spreadsheet.

The responses for some parts of this question are required on the paper provided to you.

11.

(6 points) Your company currently offers structured product based variable annuities (spVA) with a buffered and capped payout. You are assigned to value and hedge a new product that has a downside protection buffer to protect against a minimum stock price level. Specifically, the maturity payoff of the new spVA is given by:

$$\text{Payoff}_{f_T}^{\text{spVA}} = \begin{cases} S_0(1 + C) - m_0^T + S_0 & , \quad S_T \geq S_0(1 + C) \\ S_T - m_0^T + S_0 & , \quad S_0 < S_T < S_0(1 + C) \\ S_T - m_0^T + S_0 & , \quad S_T \leq S_0 \text{ and } m_0^T \geq S_0(1 - B) \\ S_T + S_0B & , \quad S_T \leq S_0 \text{ and } m_0^T < S_0(1 - B) \end{cases}$$

where B = buffer %, C = cap rate %, minimum stock price over the period of (0 to T)

$$m_0^T = \min_{0 \leq \xi \leq T} S_\xi$$

Given that the floating lookback call option (c_{fl}) and fixed lookback put option (p_{fix}) have a maturity payoff of $\max(S_T - m_0^T, 0)$ and $\max(K - m_0^T, 0)$ respectively,

(a) (3 points) Show that a portfolio of a bond (with maturity value of S_0) and the following options provides the maturity payoff of this new spVA for the ranges of $S_T \geq S_0(1 + C)$ and $S_0 < S_T < S_0(1 + C)$.

- Long a floating lookback call option
- Short a European call option with strike price $S_0(1 + C)$ (i.e., out-of-money OTM) (Hint: $S_0 \geq m_0^T$)
- Short a fixed lookback put option with strike price of $S_0(1 - B)$

11. Continued

Given the following time-0 price of the fixed lookback put option p_{fix} below:

$$p_{fix} = c_{fl}^* + Ke^{-rT} - S_0e^{-qT}$$

$$c_{fl}^* = S_0e^{-qT}N(a_1) - S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-a_1) - \min(S_0, K)e^{-rT}\left[N(a_2) - \frac{\sigma^2}{2(r-q)}e^{Y_1}N(-a_3)\right]$$

where

$$a_1 = \frac{\ln\left(\frac{S_0}{\min(S_0, K)}\right) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T}$$

$$a_3 = \frac{\ln\left(\frac{S_0}{\min(S_0, K)}\right) + (-r + q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$Y_1 = -\frac{2(r - q - \frac{\sigma^2}{2})\ln\left(\frac{S_0}{\min(S_0, K)}\right)}{\sigma^2}$$

Underlying Asset – Current Price (S_0)	100
Dividend Yield (q)	0%
Implied Volatility (σ)	20%
Term (T)	3
Risk-Free Rate (r)	5%
Buffer Rate% (B)	15%
Cap Rate% (C)	35%
Floating lookback call option (c_{fl})	30.78

The time-0 price of c_{fl} is the same as the time-0 price of c_{fl}^* with the $\min(S_0, K)$ replaced by S_0 .

- (b) (3 points) Calculate the time-0 price of the portfolio of a bond and the options specified above.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required on the paper provided to you.

12.

(8 points) Your company is reviewing different product features for a Single Premium EIA

- Choice of Indexes: S&P 500 Price Return Index ($S(T)$ is level of index at time T)
- Payoff is $S(T)^\alpha$ where $\alpha \leq 1$ is a constant
- $S(T)$ follows a geometric Brownian motion under risk-neutral measure, \mathbb{Q} with volatility, σ , and initial value $S(0) = 1$.

You would like to set the guarantee rate (g) such that the continuously compounded interest credited is 25 bps less than that of the expected value under the participation rate ($\alpha \leq 1$), to cover some potential hedging costs associated with new product. Formulaically,

$$\mathbb{E}[S(T)^\alpha]e^{-0.0025T} = e^{gT}$$

Let $r = 4\%$, $\alpha = 50\%$, $\sigma = 20\%$.

- (a) (1.5 points) Show that the guarantee rate is 1.25%.
- (b) (1 point) Derive the \mathbb{Q} -probability that the EIA credits the guaranteed rate in a single year.

You are considering implementing a cap such that the probability of $S(T)^\alpha$ exceeding the cap rate is no more than 10% in a single year. Final cap rate is rounded to the nearest whole %.

- (c) (1 point) Derive the appropriate cap rate.

12. Continued

After further discussion, your team has decided to offer the product with a cliquet design, including the cap rate. The guarantee rate is 1.25%. You are given that

$$\mathbb{E}[S(1)^{0.5} \mathbb{I}_{\{0.025 < \ln S(1) \leq .28\}}] = 0.42109$$

- (d) (3.5 points) Calculate the risk-neutral price for a 5-year cliquet EIA.
- (e) (1 point) Critique the following statement made by your analyst:

“By setting the cap rate such that the probability that $S(T)^\alpha$ exceeds cap rate is no more than 10% in a single year, we should expect to pay the cap rate approximately once every ten years.”

****END OF EXAMINATION****