Market-Consistent Risk Margins in Fair Value Loss Reserves

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Abstract

The market-consistent fair value of an insurer's unpaid claim obligations is a critical element in the determination of economic capital. It also ostensibly lies at the heart of the European Union's Solvency II Directive. Unfortunately, the Solvency-II-inspired fair value framework described in Wacek's 2008 paper, "Risk Margins in Fair Value Loss Reserves: Required Capital for Unpaid Losses and its Cost" [4] produces fair value loss reserves that are neither market-consistent nor additive. This paper introduces a minor modification to the Wacek framework that restores market consistency and additivity. The modification shifts the basis of the cost-of-capital risk margin embedded in the fair value loss reserve from an insurer's *own* solvency capital requirement with respect to its claim obligations (as prescribed by Solvency II) to the required *market-clearing* capital implied by those obligations, which depends not only on the internal characteristics of the unpaid claim portfolio but also on its correlation with the total market portfolio of unpaid claims. The modification forces the decoupling of the fair value loss reserve and required solvency capital calculations, which in the original Wacek framework (and Solvency II) are linked. The modified framework predicts that an insurer will be compensated for its cost of capital only to the extent that its reserve variability is correlated with that of the industry as a whole. As a result, an undiversified insurer may face economic and regulatory capital requirements beyond the level for which it can expect to be compensated.

Keywords: Additivity; Capital; Fair Value Loss Reserve; Market-Consistent; Risk Margin; Solvency II.

1. Introduction

This paper is a critique of, and correction to, the 2008 Wacek paper titled, "Risk Margins in Fair Value Loss Reserves: Required Capital for Unpaid Losses and its Cost" [4], which is about fair value loss reserves and the capital needed to support them. The fair value of a loss reserve is the price at which the liability for unpaid claims could be immediately and irrevocably transferred to a third party in an arms-length transaction. Because there is no active trading market for insurance claim obligations, their fair value must be estimated using principles believed to explain how such a market, if it did exist, would price such a transfer. Wacek[1] estimated the fair value loss reserve as the risk-free present value of expected unpaid claims plus a risk margin reflecting the risk-free present value of the market cost of the capital required to minimize the risk of "insolvency" due to loss reserve inadequacy during successive one-year intervals as the claims run off.[2] That working definition echoes the one given in Article 77 of the European Union's Solvency II Directive [2], for what it calls "technical provisions," a term which we will treat as synonymous with "fair value loss reserves."[3]

Using his working definition of loss reserve fair value, Wacek developed and illustrated the application of an integrated method for determining, solely from the internal characteristics of a given unpaid loss portfolio: 1) the amount of *capital* required at annual intervals throughout the runoff period to minimize the probability of insolvency over the following year due to adverse development of fair value loss reserves; 2) the implied *risk margin* equal to the risk-free present value of the market-consistent cost of that capital; and 3) the resulting *fair value estimate of the loss reserve*. His method, which calculates required capital using a value-at-risk (VaR) approach, appears capable of meeting the requirements of Solvency II, including the stipulation that risk margins be calculated separately by line of business.[4]

One criticism leveled at Wacek's fair value framework is that his method yields fair value reserves for an individual insurer that "do not add up." The sum of an insurer's fair value reserves by line of business indicated by his method generally does not equal, and usually exceeds, the same insurer's indicated total fair value reserve based directly on its total claim obligations. Likewise, the sum of individual insurer fair value reserves does not equal and usually exceeds the insurance industry's fair value reserves indicated by application of his method to the industry as a whole.[5]

There is another problem that is even more fundamental. Wacek's Solvency-II-inspired working definition of the fair value of a loss reserve is at odds with the intended meaning of fair value as a market-consistent price. The approximation of the market price of an arms-length loss

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1 All references to Wacek in this paper relate to his 2008 paper [4].

2 Insolvency in this context means that adverse loss development measured on a fair value basis exceeds the available capital assets supporting the fair value loss reserves.

3 Article 77 embodies both the working definition and the requirement that the risk margin be such that the "technical provisions" equal the amount that a third party would be expected to require to assume the unpaid claim obligations. See pp. 223–226 of the English language version [2].

4 Solvency II calls for capital to be adequate over a one-year time horizon with a probability of 99.5 percent. The effect of its stipulation that risk margins be determined by line of business is that the capital required to support loss reserves is determined at that level rather than for an insurer's loss reserve portfolio in total.

5 Because Wacek based his method on Solvency II definitions, the same criticism applies to Solvency II.

reserve transfer as a function of the capital requirement of the insurer holding the reserves relies on the assumption that the insurer's required capital is identical to the capital amount implicit in the market price. There is no basis for that assumption. An insurer's capital need is based on the volatility and other characteristics of its own loss reserve portfolio. In contrast, the capital requirement underlying the market price for the transfer of those reserves depends not only on the internal characteristics of the subject loss reserve portfolio but also on its correlation with the other loss reserves with which the subject reserves could be commingled after the transfer. Only under rare circumstances would an individual insurer's and the market-consistent capital requirements be the same. As a result, if our intention is to calculate fair value loss reserves on the basis of market-consistent cost-of-capital risk margins, we cannot reliably use an insurer's own capital requirement to determine that fair value.[6]

The purpose of this paper is to address and resolve these two problems. Fortunately, it is possible to do so by the elaboration and restatement of the formula for a single key variable within the Wacek framework. As we will show, that modification leads to the ability to disaggregate the total market capital and risk margin (and thus the fair value reserve) into additive components. Most of Wacek's algebraic framework remains intact, though the conceptual interpretation of some of the formulas changes. Our solution does come at a cost. We have to decouple the calculation of the capital underlying the market-consistent cost of capital used to determine the risk margin and fair value loss reserve from the calculation of the capital required to minimize an insurer's insolvency risk due to adverse fair value loss reserve development. That is not a significant problem, but it does mean that after determining an insurer's market-consistent fair value loss reserves using the modified Wacek method, we have to circle back and separately calculate the solvency capital required to support those reserves.

The paper has five sections, the first of which is this Introduction, and a pair of appendices containing some technical backup. In Section 2 we outline how, in a basic VaR-based capital framework within a competitive market, total capital can be disaggregated into additive components using covariance relationships. Section 3 applies the concepts introduced in Section 2 to the more complex situation surrounding fair value loss reserves, where the capital underlying fair value loss reserves must be held for multiple years. It is in this section that we present our modification to the original Wacek framework. In Section 4 we illustrate the application of our revised approach to a simplified insurance industry comprising three lines of business and three accident years, as well as to a single monoline insurer. In Section 5 we summarize the key points of the paper and point to potential further research and development. A complete set of abbreviations and notations appears after the two appendices followed by a list of references.

[6] A further clue that the fair value loss reserves and risk margins produced by the Wacek [4] framework are not market-consistent is their non-additivity. In his 1991 paper, "Premium Calculation Implications of Reinsurance without Arbitrage," Venter [3] used a no-arbitrage argument to show that market-consistent risk margins must be additive; if not, they will not be market-consistent.

2. Additive Capital in Basic Value-at-Risk Framework

In very basic terms and ignoring present value and time horizon considerations, a VaR-based capital adequacy standard targeting an α probability of meeting all obligations yields a capital requirement C_T^R with respect to risk *T* equal to the amount by which the cost of the adverse outcome at the α-percentile $VaR_\alpha(T)$ exceeds the assets held to fund the expected outcome $E(T)$:

$$C_T^R = VaR_\alpha(T) - E(T). \tag{2.1}$$

If the standard deviation of risk outcomes σ_T is finite, which is a reasonable assumption for the practical applications we are interested in, that capital amount can be expressed in terms of the standard deviation as:

$$C_T^R = NSD_T(\alpha) \cdot \sigma_T \tag{2.2}$$

where $NSD_T(\alpha)$ is a mnemonic for "number of standard deviations" corresponding to α.

If risk *T* is itself the sum of multiple risk components, *m* in number, such that the outcome $x_T = \sum^m x_j$, then the standard deviation of risk *T* outcomes in terms of its component parts is given by:

$$\sigma_T = \rho_{1T} \cdot \sigma_1 + \rho_{2T} \cdot \sigma_2 + \rho_{3T} \cdot \sigma_3 + \cdots + \rho_{mT} \cdot \sigma_m \tag{2.3}$$

where ρ_{jT} represents the correlation coefficient (a standardized measure of covariance) between risk component *j* and the total risk *T* for $1 \le j \le m$. In words, Formula (2.3) says that the total standard deviation is the correlation-weighted sum of the standard deviations of the components comprising the total.[7]

Multiplying both sides of Formula (2.3) by $NSD_T(\alpha)$, we see that the total capital requirement, as a function of σ_T, can be expressed in terms of its risk component parts as:

$$NSD_T(\alpha) \cdot \sigma_T = \sum_j^m NSD_T(\alpha) \cdot \rho_{jT} \cdot \sigma_j . \tag{2.4}$$

The right side of Formula (2.4) implies a contribution C_j^T from each risk component *j* to the total capital requirement of:

$$C_j^T = NSD_T(\alpha) \cdot \rho_{jT} \cdot \sigma_j \tag{2.5}$$

for each $1 \le j \le m$.

[7] Readers unfamiliar with Formula (2.3) can refer to Appendix A for a derivation.

Note the contrast of C_j^T with the capital requirement C_j^R for each risk component *j in isolation* of:

$$C_j^R = NSD_j(\alpha) \cdot \sigma_j \tag{2.6}$$

for each $1 \le j \le m$.[8]

If we think of *T* as the total market comprising *m* insurers, then insurer *j* contributes $C_j^T = NSD_T(\alpha) \cdot \rho_{jT} \cdot \sigma_j$ to the market's total capital requirement but needs stand-alone capital of $C_j^R = NSD_j(\alpha) \cdot \sigma_j$ to minimize its own risk of insolvency.[9] In a competitive market for loss reserves, we would expect the market-clearing capital requirement for each insurer's loss reserve portfolio to reflect its contribution $NSD_T(\alpha) \cdot \rho_{jT} \cdot \sigma_j$ to the hypothetical capital requirement for the total market portfolio.[10] In that case, the cost-of-capital risk margin in each insurer's fair value loss reserves would be based on that market-clearing capital requirement and not on its own solvency capital requirement.

This dynamic originates in the competitive market pricing of insurance policies. The market-clearing premium reflects the present value of expected claims plus a risk margin reflecting the market cost of capital (i.e., the fair value of expected claims at policy inception) and a market-clearing provision for expenses. The size of the risk margin and the expense provision is determined by market risk considerations and not by the capital requirement implied by an insurer's own portfolio. An insurer seeking to recover its own higher cost of capital and/or expenses will typically find itself priced out of the market. In order to earn an adequate return on capital, such an insurer faces competitive pressure to structure its portfolio in such a way as to bring its required solvency capital and expenses into line with market-clearing norms.

The competitive market pricing of insurance premiums lends support to our expectation that the risk margin in fair value reserves be based on the market-clearing capital requirement rather than on an insurer's own solvency capital requirement. The risk margin in a fair value loss

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8 It is very important to note that $NSD_T(\alpha) \ne NSD_j(\alpha)$ except under special circumstances (e.g., x_i normally distributed for all $1 \le j \le m$, which implies that x_T is also normal). That means that the total capital requirement cannot reliably be expressed as the sum of correlation-adjusted individual risk source $VaR_\alpha(x_j)$ amounts: $C_T^R \ne \sum_j^m \rho_{jT} \cdot C_j^R$ unless x_j is normally distributed for all $1 \le j \le m$ (and possibly other special cases unknown to the author).

9 Note that this characterization of the insurer's contribution to the market capital need overstates the market requirement by ignoring the fact that its contribution to market loss is limited to its own capital. However, for plausible values of $\alpha \ge 99\%$, we believe the overstatement is very small.

10 While there is no actual "capital requirement for the total market," competition among the large diversified insurers that are in the best position to assume loss reserve portfolios will drive their portfolios toward the characteristics of the total market. It is possible that less-than-perfect competition might leave the key capital components $NSD(\alpha)$ and/or σ of the largest, most diversified insurers slightly higher than those of the total market as a monopoly. If that is a concern, it could be addressed by using a slightly higher value of α in calculating the hypothetical capital for the total market portfolio.

reserve originates as a component of the premiums collected for the policies producing the claim obligation. As the claim obligation runs off, we would expect that the risk margin, now embedded in the fair value loss reserve, would continue to reflect the cost of market-clearing capital.

We now turn to a more rigorous application of the concepts introduced in this section to the fair value of a claim obligation within the Wacek framework, which is the subject of Section 3.

3. Additive Capital in (Modified) Wacek Framework

Wacek defined the required capital C_n^R to support a fair value loss reserve $T(L_n)$ with respect to expected unpaid claim obligation L_n as of time *n* as:

$$C_n^R = v_1 \cdot VaR_\alpha(t_{n+1}) - T(L_n), \qquad (2.16W)[11]$$

where $v_1 = (1+r_1)^{-1}$ is the one-year risk-free discount factor as of time *n*; t_{n+1} represents the random variable, defined as of time *n*, for the fair value of the one-year hindsight estimate of L_n at time $n+1$; and $VaR_\alpha(t_{n+1})$ is the value-at-risk with respect to t_{n+1} at the α confidence level.

Formula (2.16W) is a rigorous expression, incorporating present value and time horizon considerations, of Formula (2.1) and its accompanying statement that “a VaR-based capital adequacy standard targeting a α probability of meeting all obligations yields a capital requirement with respect to risk *T* equal to the amount by which the cost of the adverse outcome at the α-percentile exceeds the assets held for the purpose of funding the expected outcome,” where now “risk *T*” refers to the risk associated with the fair value of the claim obligation as of time *n*.

Wacek determined the fair value risk margin R'_n with respect to the claim obligation L_n as of time *n* to be:

$$R'_n = v_1 \cdot ((roe_{PT} - r_1) \cdot C_n^R + R'_{n+1}) \qquad (2.6W)$$

where r_1 is the risk-free rate for one-year money as of time *n*, $v_1 = (1+r_1)^{-1}$; roe_{PT} is the annual required pretax return on equity (i.e., capital); and R'_{n+1} is the fair value risk margin with respect to the expected unpaid claim obligation L_{n+1} as of time $n+1$.

He also derived an alternative formula for required capital as of time *n* in terms of the characteristics one year out of the underlying claim obligation itself (instead of its fair value) and the fair value risk margin:

$$C_n^R = \frac{F_{n+1} + f_{n+1} \cdot R'_{n+1}}{1 + roe_{PT}} \qquad (2.20W)$$

where F_{n+1} and f_{n+1} are functions of the underlying claim obligation.[12]

[11] Where we are merely recapping formulas presented in Wacek [4], we use the formula numbers from that paper with “W” appended at the end. Note that Wacek focused on a single aggregate claim source and omitted subscripts such as our *j* and *T*.

[12] F_{n+1} is the additional amount of assets needed at time $n+1$ to bring present value loss funding up to the α confidence level. f_{n+1} is the fraction by which the time $n+1$ present value unpaid losses embedded in the one-year hindsight estimate at the α confidence level exceed the expected time $n+1$ present value unpaid loss amount.

Wacek showed that, assuming all claims underlying L_n are paid by time $n+k$, it is possible to determine C_n^R and R_n' by working backward recursively from time $n+k$. However, if we are interested only in the fair value risk margin, it is possible to express R_n' directly as:

$$
\begin{aligned}
R_n' &= v_1 \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} \cdot F_{n+1} \\
&+ v_2^2 \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} \cdot F_{n+2} \\
&+ v_3^3 \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{2:1}}{1+roe_{PT}} \cdot F_{n+3} \\
&+ \cdots \\
&+ v_k^k \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdots (f_{n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{k-1:1}}{1+roe_{PT}} \cdot F_{n+k}
\end{aligned}
\tag{3.1}
$$

where $r_{i:1}$ is the *i*-year forward rate risk-free rate for one-year money as of time *n* (for $0 \le i \le k-1$) with $r_{0:1}$ denoted simply as r_1.[13]

Each of the terms of the form F_{n+i+1} in the numerator of Formula (3.1) represents the additional amount of assets needed at time $n+i+1$ (for $0 \le i \le k-1$) to bring present value loss-only funding (i.e., with no risk margin) of the claim obligation L_{n+i} up to the α confidence level, which Wacek expressed as Formula (2.24W):

$$
\begin{aligned}
F_{n+i+1} &= VaR_\alpha(PV(h_{n+i+1})) - PV(L_{n+i}) \cdot (1+r_{i:1}) \\
&= VaR_\alpha(PV(h_{n+i+1})) - E(PV(h_{n+i+1}))
\end{aligned}
\tag{2.24W}
$$

for $0 \le i \le k-1$, and where h_{n+i+1} represents the random variable, defined as of time $n+i$ years, for the one-year hindsight estimate of L_{n+i} at time $n+i+1$ years; $VaR_\alpha(PV(h_{n+i+1}))$ is the α confidence level value-at-risk with respect to the risk-free present value at time $n+i+1$ of h_{n+i+1}; $PV(L_{n+i})$ is the risk-free present value at time $n+i$ of L_{n+i}.

Put another way, F_{n+i+1} is the future value (one year out) of the time $n+i$ capital required to support the pure claim obligation at time $n+i+1$. In that respect it is analogous to the basic capital requirement C_T^R defined in Formulas (2.1) and (2.2). Like C_T^R, F_{n+i+1} can be expressed in terms of the standard deviation of the prospective risk outcomes:

$$
F_{n+i+1} = NSD_{n+i+1}(\alpha) \cdot \sigma_{n+i+1} \tag{3.2}
$$

where σ_{n+i+1} is the standard deviation of $PV(h_{n+i+1})$ and $NSD_{n+i+1}(\alpha)$ is the number of standard deviations σ_{n+i+1} corresponding to α.[14]

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[13] See Appendix B for the derivation Formula (3.1).

The alternative characterization of F_{n+i+1} provided by Formula (3.2) is the key to unlocking both additivity and market consistency in fair value risk margins. If the claim obligation $L_{T,\,n+i}$ is the aggregated industry total of claims from *m* sources, then $L_{T,\,n+i} = \sum_j^m L_{j,\,n+i}$ and, using Formula (2.4), we can express the corresponding total industry α confidence level funding requirement $F_{T,\,n+i+1}$ in terms of the respective contributions of the *m* component claim sources as follows:

$$F_{T,\,n+i+1} = NSD_{T,\,n+i+1}(\alpha) \cdot \sum_j^m \rho_{j,\,n+i+1} \cdot \sigma_{j,\,n+i+1} \tag{3.3}$$

$$= \sum_j^m F^T_{j,\,n+i+1} \tag{3.4}$$

where

$$F^T_{j,\,n+i+1} = NSD_{T,\,n+i+1}(\alpha) \cdot \rho_{j,\,n+i+1} \cdot \sigma_{j,\,n+i+1}\,. \tag{3.5}$$

Formulas (3.2) through (3.5) together represent our crucial modification to the Wacek framework. We will show that, by using $F^T_{j,\,n+i+1}$ as defined by Formula (3.5) in place of F_{n+i+1} in the original Wacek framework, we can calculate the risk margin for any claim source *j* such that the sum of the risk margins from all *m* claim sources equals the risk margin calculated directly at the total market level. In other words, the risk margins are additive.

If we add a subscript *T* to each of R'_n, f_{n+i+1} and F_{n+i+1} in Formula (3.1) and substitute $F_{T,\,n+i+1} = \sum_j^m F^T_{j,\,n+i+1}$ from Formula (3.4), we obtain the following formula for the risk margin $R'_{T,\,n}$ for the total industry claim obligation $L_{T,\,n}$:

$$\begin{aligned}
R'_{T,\,n} &= v_1 \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} \cdot \sum_j^m F^T_{j,\,n+1} \\
&+ v_2^2 \cdot \left(f_{T,\,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} \cdot \sum_j^m F^T_{j,\,n+2} \\
&+ v_3^3 \cdot \left(f_{T,\,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \left(f_{T,\,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{2:1}}{1 + roe_{PT}} \cdot \sum_j^m F^T_{j,\,n+3} \\
&+ \cdots \\
&+ v_k^k \cdot \left(f_{T,\,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \left(f_{T,\,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1\right) \cdots \\
&\qquad \cdots \left(f_{T,\,n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{k-1:1}}{1 + roe_{PT}} \cdot \sum_j^m F^T_{j,\,n+k}
\end{aligned} \tag{3.6}$$

[14] Formula (3.2) is defined only if σ_{n+i+1} is finite.

It is clear from Formula (3.6) that the total industry risk margin $R'_{T,n}$ as of time n is the sum of distinct contributions from each of the m claim sources:

$$R'_{T,n} = \sum_{j}^{m} R'_{j,n} , \tag{3.7}$$

where

$$\begin{aligned}
R'_{j,n} &= v_1 \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} \cdot F^T_{j,n+1} \\
&+ v_2^2 \cdot \left(f_{T,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} \cdot F^T_{j,n+2} \\
&+ v_3^3 \cdot \left(f_{T,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \left(f_{T,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{2:1}}{1 + roe_{PT}} \cdot F^T_{j,n+3} \\
&+ \cdots \\
&+ v_k^k \cdot \left(f_{T,n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1\right) \cdot \left(f_{T,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1\right) \cdots \\
&\qquad \cdots \left(f_{T,n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1 + roe_{PT}} + 1\right) \cdot \frac{roe_{PT} - r_{k-1:1}}{1 + roe_{PT}} \cdot F^T_{j,n+k}
\end{aligned} \tag{3.8}$$

Alternatively, Wacek's original recursive framework can also be used to determine $R'_{j,n}$ and $C^T_{j,n}$, where $C^T_{j,n}$ represents *market-clearing capital* (which is not necessarily the same as solvency capital). All that is required is the use of the following "*j*-level" formulas for $R'_{j,n+i}$ and $C^T_{j,n+i}$ instead of Wacek Formulas (2.7W) and (2.23W):[15]

$$R'_{j,n+i} = v_{i:1} \cdot \left((roe_{PT} - r_{i:1}) \cdot C^T_{j,n+i} + R'_{j,n+i+1}\right) \tag{3.9}$$

$$C^T_{j,n+i} = \frac{F^T_{j,n+i+1} + f_{T,n+i+1} \cdot R'_{j,n+i+1}}{1 + roe_{PT}} . \tag{3.10}$$

If it makes sense to talk about solvency at the level of the subset of total industry claims represented by claim source *j* (such as when *j* represents the total claim obligations of a particular insurer), then the required *solvency capital* $C^R_{j,n}$ as of time *n* is given by:

$$C^R_{j,n} = \frac{F_{j,n+1} + f_{j,n+1} \cdot R'_{j,n+1}}{1 + roe_{PT}} \tag{3.11}$$

[15] Note that, if *T* is substituted for *j* in Formulas (3.9) and (3.10), the formulas also produce correct results for the risk margin and capital at the level of the total claim obligation.

where $F_{j,n+1}$ and $f_{j,n+1}$ are functions of the *underlying claim obligation j* defined according to Wacek Formulas (2.21W) and (2.22W) and $R'_{j,\,n+1}$ is the *market-clearing* risk margin defined by recursive Formula (3.9) or the following direct formula:[16]

$$\begin{aligned}
&R'_{j,\,n+1} = v_{1:1} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} \cdot F^{T}_{j,\,n+2} \\
&+ v_{1:1} \cdot v_{2:1} \cdot (f_{T,\,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{2:1}}{1 + roe_{PT}} \cdot F^{T}_{j,\,n+3} \\
&+ \cdots \\
&+ v_{1:1} \cdot v_{2:1} \cdots v_{k-1:1} \cdot (f_{T,\,n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1 + roe_{PT}} + 1) \cdots (f_{T,\,n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1 + roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{k-1:1}}{1 + roe_{PT}} \cdot F^{T}_{j,\,n+k}
\end{aligned} \tag{3.12}$$

To recap the key points of this section, by substituting $F^{T}_{j,\,n+i+1}$ as defined by Formula (3.5) for F_{n+i+1} in Wacek's original formulas and applying his recursive procedure using Formulas (3.9) and (3.10), we can obtain additive risk margins and fair value reserves. (An important caveat is that the capital calculated from the recursive procedure must now be interpreted as *market-clearing* rather than *solvency* capital.) Because market-clearing capital itself is not needed, we can also determine the risk margin directly by using the non-recursive Formula (3.12). In the modified framework, solvency capital is no longer a byproduct of the recursive procedure and must now be determined directly using Formula (3.11).

We find it helpful to test any theoretical framework with realistic numbers to see what it would look like in practice. In Section 4, using the formulas we have presented in Section 3, we illustrate the calculation of 1) risk margins by line and year for a simplified insurance industry, and 2) the total risk margin, required solvency capital and expected pretax return on equity for a monoline insurer operating within that simplified industry.

[16] See Appendix B for the derivation of this relationship. Formula (3.12) presented here differs from Formula (B.4) derived in Appendix B only in its references to *T* and *j*, which are omitted in Formula (B.4).

4. Illlustration of Modified Wacek Framework

In this section we illustrate our modified Wacek framework for calculating market-consistent risk margins in fair value reserves presented in Section 3. For the sake of illustration, we assume that the insurance industry comprises three lines of business A, B and C and that within each of those lines all claims can be expected to be paid within four years of inception.[17] Table A summarizes the undiscounted "industry" loss reserves as of Dec. 31, 2009 for these lines (scaled to an overall total of $10,000 for ease of presentation) arising from accident years 2007, 2008 and 2009.[18]

TABLE A
Undiscounted Loss Reserves
Industry Total
as of December 31, 2009

Acc Year	Line A	Line B	Line C	Total
2007	$305	$461	$1,449	$2,215
2008	389	645	2,024	3,058
2009	1,535	790	2,402	4,727
Total	$2,228	$1,896	$5,876	$10,000

Our aim is to determine the risk margin and fair value loss reserve for each reserve cell in Table A. We assume that the Dec. 31, 2009 risk-free yield curve implies a spot rate r_1 for one-year money of 3.34 percent, a one-year forward rate $r_{1:1}$ for one-year money of 2.76 percent and a two-year forward rate $r_{2:1}$ for one-year money of 3.11 percent.[19] We assume that the pretax required return on equity roe_{PT} is a constant spread of 12.5 percent over the anticipated risk-free rate, which results in the following sequence of annual pretax return requirements during the runoff period: $roe_{PT,\,0} = 15.84\%$, $roe_{PT,\,1} = 15.26\%$ and $roe_{PT,\,2} = 15.61\%$.

Table B-1 shows the risk-free present value $PV(L_{j,\,2009})$, as of Dec. 31, 2009, of the loss reserves tabulated in Table A together with the standard deviation $\sigma_{j,\,2010}$ of prospective one-year loss development for each reserve cell and the correlation coefficient $\rho_{jT,\,2010}$ of that development with the industry total. Both the standard deviations and correlation coefficients were calculated with respect to present values at the end of the one-year development period, i.e., as of the end of 2010. Note that the standard deviations, which are expressed as ratios to the expected present value one-year hindsight reserves at year-end 2010,[20] tend to be higher for the

[17] The illustration, intended to be reasonably realistic, is based on an analysis of recent industry Schedule P data of three lines of business. However, in a departure from realism, all claims were assumed to be paid within four years of inception, at which point any unpaid claims were simply treated as paid.

[18] Slight differences between row and column sums and the displayed subtotals are due to rounding.

[19] For further background on the relationship between spot and forward rates, see Appendix A of [4].

[20] Bear in mind that these expected present values as of year-end 2010 are higher than the year-end 2009 present value loss reserves by a factor of $1 + r_1 = 1.0334$. The comparable factors in Tables B-2 and B-3 are 1.0276 and 1.0311, respectively.

older accident years. To the extent that this subset of industry data is representative, it appears that loss reserve development becomes more variable as claims approach settlement. Also note that Line A correlation coefficients are lower than those of Lines B and C. Everything else being equal, that implies that Line A risk margins will tend to be smaller than those for Lines B and C.

The bottom part of Table B-1 shows the contribution $F^{T}_{j,\,2010}$ of each reserve cell *j* to the total industry 2010 loss funding requirement $F_{T,\,2010}$ of $1,542 at the $\alpha = 99.5\%$ level. Each $F^{T}_{j,\,2010}$ was calculated using Formula (3.5), with the value of $NSD_{T,\,2010}(99.5\%) = 2.71$ implied by $F_{T,\,2010}$.[21] For example, the calendar year 2010 contribution of the reserve cell corresponding to Line C and accident year 2009 to the total industry funding requirement is $F^{T}_{2009\ C,\,2010} = 2.71 \times 0.77 \times (6.5\% \times \$2{,}325 \times 1.0334) = \$326$. The total Line A contribution (with respect to accident years 2007 through 2009 combined) to the total 2010 industry funding requirement is $F^{T}_{A,\,2010} = 2.71 \times 0.67 \times (8.4\% \times \$2{,}130 \times 1.0334) = \$336$. Alternatively, as an illustration of additivity, the total Line A contribution $F^{T}_{A,\,2010}$ can also be determined by summing the *F* values for the three Line A reserve cells corresponding to accident years 2007 through 2009: $F^{T}_{2009,\,2010} = \$32 + \$48 + \$256 = \$336$.

Tables B-2 and B-3 are compilations of the same statistics shown in Table B-1 but valued from the vantage points of Dec. 31, 2010 and Dec. 31, 2011, respectively. The expected present value loss reserves for accident year 2007 are zero in Table B-2, reflecting our assumption that all claims are paid within four years of accident year inception. For the same reason, the present value reserves shown in Table B-3 for both accident years 2007 and 2008 are zero. The bottom portion of each of Tables B-2 and B-3 shows the expected contributions $F^{T}_{j,\,2011}$ and $F^{T}_{j,\,2012}$ of each reserve cell *j* to the total industry loss funding requirements at the 99.5% confidence level in calendar years 2011 and 2012 of $1,048 and $713, respectively. The contribution in calendar year 2011 for the Line B/accident year 2008 reserve cell (in Table B-2) is $F^{T}_{2008\ B,\,2011} = 2.79 \times 0.7 \times (14.5\% \times \$435 \times 1.0276) = \$127$. The expected calendar year 2012 contribution for the Line A/accident year 2009 reserve cell (in Table B-3) is $F^{T}_{2009\ A,\,2012} = 3.25 \times 0.56 \times (11.3\% \times \$296 \times 1.0311) = \$63$.

[21] Note that the *F* values displayed in Table B-1 were calculated with full precision inputs and may vary slightly from values calculated using the rounded input values shown in the table.

TABLE B
Key Loss Reserve Development Statistics
Industry Total
as of December 31, 2009

Present Value Loss Reserves: $PV(L_{j,\,2009})$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$279	$421	$1,324	$2,023
2008	366	607	1,906	2,879
2009	1,485	764	2,325	4,574
Total	$2,130	$1,793	$5,554	$9,477
Standard Deviation of 1-Yr Development: $\sigma_{j,\,2010}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	11.3%	14.5%	10.3%	9.9%
2008	9.6%	10.8%	6.5%	7.1%
2009	10.3%	7.4%	6.5%	6.8%
Total	8.4%	8.1%	5.8%	5.8%
1-Yr Development Correlation With Total: $\rho_{jT,\,2010}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	0.37	0.52	0.65	0.66
2008	0.48	0.80	0.79	0.83
2009	0.60	0.75	0.77	0.80
Total	0.67	0.87	0.95	1.00
Contribution to 1-Yr Funding Need: $F^{T}_{j,\,2010}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$32	$89	$249	$370
2008	48	147	276	471
2009	256	119	326	701
Total	$336	$356	$851	$1,542
r_1 =3.34%		$NSD_{T,\,2010}(99.5\%)$		2.71

TABLE B-2
Key Loss Reserve Development Statistics
Industry Total
as of December 31, 2010

Expected Present Value Loss Reserves: $PV(L_{j,\,2010})$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$0	$0	$0	$0
2008	288	435	1,368	2,091
2009	378	628	1,970	2,975
Total	$666	$1,063	$3,337	$5,066
Standard Deviation of 1-Yr Development: $\sigma_{j,\,2011}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007				
2008	11.3%	14.5%	10.3%	9.9%
2009	9.6%	10.8%	6.5%	7.1%
Total	8.6%	11.0%	7.1%	7.2%
1-Yr Development Correlation with Total: $\rho_{jT,\,2011}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007				
2008	0.45	0.70	0.87	0.87
2009	0.47	0.87	0.84	0.88
Total	0.56	0.88	0.98	1.00
Contribution to 1-Yr Funding Need: $F^{T}_{j,\,2011}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$0	$0	$0	$0
2008	42	127	351	520
2009	50	168	310	528
Total	$92	$295	$661	$1,048
$r_{1:1}$ =2.76%		$NSD_{T,\,2011}(99.5\%)$		2.79

TABLE B-3
Key Loss Reserve Development Statistics
Industry Total
as of December 31, 2011

Expected Present Value Loss Reserves: $PV(L_{j,\,2011})$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$0	$0	$0	$0
2008	0	0	0	0
2009	296	447	1,406	2,149
Total	$296	$447	$1,406	$2,149
Standard Deviation of 1-Yr Development: $\sigma_{j,\,2012}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007				
2008				
2009	11.3%	14.5%	10.3%	9.9%
Total	11.3%	14.5%	10.3%	9.9%
1-Yr Development Correlation with Total: $\rho_{jT,\,2012}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007				
2008				
2009	0.56	0.80	0.98	1.00
Total	0.56	0.80	0.98	1.00
Contribution to 1-Yr Funding Need: $F^{T}_{j,\,2012}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$0	$0	$0	$0
2008	0	0	0	0
2009	63	174	476	713
Total	$63	$174	$476	$713
$r_{2:1}$ =3.11%		$NSD_{T,\,2012}(99.5\%)$		3.25

4.1 Industry Level Risk Margins

We know from Formula (3.8) that the Dec. 31, 2009 required risk margin $R'_{j,\,2009}$ for reserve cell *j* is a function of the funding contributions $F^T_{j,\,2010}, F^T_{j,\,2011}$ and $F^T_{j,\,2012}$ in calendar years 2010, 2011 and 2012. Formula (3.8) can be expressed succinctly as:

$$R'_{j,\,n} = coeff_{n,1} \cdot F^T_{j,\,n+1} + coeff_{n,2} \cdot F^T_{j,\,n+2} + coeff_{n,3} \cdot F^T_{j,\,n+3} + \cdots + coeff_{n,k} \cdot F^T_{j,\,n+k} \qquad (4.1)$$

where $coeff_{2009,1}$, $coeff_{2009,2}$ and $coeff_{2009,3}$ are as tabulated in Table C. Their values in this illustration are shown in the rightmost column, and are based on the risk-free yield curve, the required pretax return on equity assumptions given at the beginning of this section, and $f_{T,\,2010} = 0.22$ and $f_{T,\,2011} = 0.30$.[22]

TABLE C
Coefficients of $F^T_{j,\,n+i+1}$ in Formula (4.1) for $R'_{j,n}$
($n = 2009$,)

	Algebraic Expression	Value
$coeff_{2009,1}$	$v_1 \cdot \dfrac{roe_{PT,\,0} - r_1}{1 + roe_{PT,\,0}}$	0.1044
$coeff_{2009,2}$	$v_2^2 \cdot (f_{T,\,2010} \cdot \dfrac{roe_{PT,\,0} - r_1}{1 + roe_{PT,\,0}} + 1) \cdot \dfrac{roe_{PT,\,1} - r_{1:1}}{1 + roe_{PT,\,1}}$	0.1046
$coeff_{2009,3}$	$v_3^3 \cdot (f_{T,\,2010} \cdot \dfrac{roe_{PT,\,0} - r_1}{1 + roe_{PT,\,0}} + 1) \cdot (f_{T,\,2011} \cdot \dfrac{roe_{PT,\,1} - r_{1:1}}{1 + roe_{PT,\,1}} + 1) \cdot \dfrac{roe_{PT}}{1 +}$	0.1044

We are now in a position to determine the risk margin $R'_{j,\,2009}$ in Dec. 31, 2009 fair value loss reserves for each reserve cell *j* and in total. The upper portion of Table D displays the values of the inputs required by Formula (4.1), while the bottom portion shows the resulting risk margins as of Dec. 31, 2009 by reserve cell and in total. The total risk margin for all lines and accident years, for example, is the following result from the application of Formula (4.1): $\$1{,}542 \times 0.1044 + \$1{,}048 \times 0.1046 + \$713 \times 0.1044 = \$345$.

[22] $f_{T,\,2010}$ is the fraction by which the year-end 2010 total industry present value unpaid losses embedded in the one-year hindsight estimate of 2009 present value reserves at the α confidence level exceed the expected year-end 2010 present value unpaid loss amount. $f_{T,\,2011}$ is the same fraction shifted one year out in the runoff period. See Formula (2.22) in Wacek [4] for the general definition in algebraic terms.

TABLE D
Summary of Risk Mark Calculation
Industry Total
As of December 31, 2009

Contribution to 1-Yr Funding Need in 2010: $F^{T}_{j,\,2010}$					
Acc Year	**Line A**	**Line B**	**Line C**	**Total**	$coeff_{2009,1}$
2007	$32	$89	$249	$370	
2008	48	147	276	471	0.1044
2009	256	119	326	701	
Total	$336	$356	$851	$1,542	

Contribution to 1-Yr Funding Need in 2011: $F^{T}_{j,\,2011}$					
Acc Year	Line A	Line B	Line C	Total	$coeff_{2009,2}$
2007	$0	$0	$0	$0	
2008	42	127	351	520	0.1046
2009	50	168	310	528	
Total	$92	$295	$661	$1,048	

Contribution to 1-Yr Funding Need in 2012: $F^{T}_{j,\,2012}$					
Acc Year	**Line A**	**Line B**	**Line C**	**Total**	$coeff_{2009,3}$
2007	$0	$0	$0	$0	
2008	0	0	0	0	0.1044
2009	63	174	476	713	
Total	$63	$174	$476	$713	

Risk Margin in Fair Value Loss Reserves: $R'_{j,\,2009}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$3	$9	$26	$39
2008	9	29	65	103
2009	38	48	116	203
Total	$51	$86	$208	$345

4.2 Industry Level Fair Value Loss Reserves

The fair value loss reserve $T(L_{j,\,2009})$ as of Dec. 31, 2009 for each industry reserve cell is the sum of the risk-free present value of that reserve cell's claim obligation plus the corresponding risk margin. Table E summarizes the key ingredients that make up the fair value loss reserves in this illustration. The undiscounted industry loss reserves are recapped at the top of the table followed by the risk-free present value of those reserves and the corresponding risk margins. The corresponding fair value loss reserves appear in the second reserve cell array from the bottom.

The industry grand total fair value reserve amount, for example, is shown as $9,822, which is the sum of the present value reserve of $9,477 and risk margin of $345. The fair value of the accident year 2009 industry claim obligation for Line C of $2,441 is the sum of the present value reserve of $2,325 and risk margin of $116. Because present value loss reserves and risk margins are additive, we obtain the same total (or subtotal) fair value loss reserve amounts irrespective of whether we sum the fair values by reserve cell or first add up the present values and risk margins and sum the results.

The array at the bottom of Table E shows the ratios of the fair value industry loss reserves to the undiscounted industry loss reserves by reserve cell. The ratios are highest for accident year 2009 reserves and lowest for accident year 2007 reserves. For accident year 2009 the fair value reserves for lines B and C as well as for the total exceed the undiscounted loss reserves.

If we focus on the risk margin components, the variation by line of business and accident is even more evident. Table F shows the ratio of risk margin to present value loss reserve by reserve cell. The risk margin associated with the accident year 2009 claim obligation of Line B, 6.3 percent of the present value reserve, is more than five times the risk margin for the accident year 2007 claim obligation of Line A (1.2 percent). We see that the accident year 2007 risk margins by line are lower than accident year 2008 risk margins, and the latter are generally lower than the accident year 2009 risk margins. The reason for that accident year pattern is that capital has to be held for only one year with respect to accident year 2007, while accident year 2008 will require capital for two years, and accident year 2009 will need it for three years. There is also significant variation between lines. For example, for accident year 2009 the Line A risk margin of 2.6 percent is less than half the Line B risk margin of 6.3 percent, a difference driven by much larger *F* values for Line B relative to expected reserves in the second and third years of the runoff period, which, in turn, can be traced back to higher Line B correlation and volatility.

TABLE E
Summary of Loss Reserve Statistics
Industry Total
As of December 31, 2009

Undiscounted Loss Reserves: $L_{j,\,2009}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$305	$461	$1,449	$2,215
2008	389	645	2,024	3,058
2009	1,535	790	2,402	4,727
Total	2,228	1,896	5,876	$10,000
Present Value Loss Reserves: $PV(L_{j,\,2009})$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$279	$421	$1,324	$2,023
2008	366	607	1,906	2,879
2009	1,485	764	2,325	4,574
Total	2,130	1,793	5,554	$9,477
Risk Margin in Fair Value Loss Reserves: $R'_{j,\,2009}$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$3	$9	$26	$39
2008	9	29	65	103
2009	38	48	116	203
Total	$51	$86	$208	$345
Fair Value Loss Reserves: $T(L_{j,\,2009})$				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	$282	$430	$1,350	$2,062
2008	375	636	1,971	2,983
2009	1,524	813	2,441	4,777
Total	$2,181	$1,879	$5,762	$9,822
Ratio of Fair Value to Undiscounted Loss Reserves				
Acc Year	**Line A**	**Line B**	**Line C**	**Total**
2007	0.924	0.933	0.931	0.931
2008	0.966	0.986	0.974	0.976
2009	0.993	1.029	1.016	1.011
Total	0.979	0.991	0.981	0.982

TABLE F
Ratio of Risk Margin to Present Value Loss Reserve
Industry Total
As of December 31, 2009

Acc Year	Line A	Line B	Line C	Total
2007	1.2%	2.2%	2.0%	1.9%
2008	2.6%	4.7%	3.4%	3.6%
2009	2.6%	6.3%	5.0%	4.4%
Total	2.4%	4.8%	3.7%	3.6%

4.3 Insurer ABC Risk Margin, Fair Value Loss Reserve and Solvency Capital

Suppose insurer ABC is a monoline underwriter of Line B business and that it holds 10 percent of the Dec. 31, 2009 industry Line B claim obligations arising from accident years 2007 through 2009. We assume that the correlation of ABC's reserve development by accident year reserve cell with the industry total is the same as that we observed for the Line B at the industry level. However, we expect its one-year reserve development to be about 15 percent more volatile as measured by the standard deviation. In every other respect ABC's loss reserves behave like the industry's Line B reserves.

As a slice of the industry, insurer ABC's risk margins can be calculated using Formula (4.1) in exactly the same way we used it to determine risk margins for the industry level reserve cells. However, because we have assumed certain relationships between ABC and the industry, we will take advantage of some shortcuts in order to get to our point more quickly.

ABC's total risk margin $R'_{ABC\,,\,2009}$ as of Dec. 31, 2009 is \$9.91 (10 percent of the industry risk margin for Line B times 1.15). Given the total present value loss reserve of \$179.28 (10 percent of \$1,793), insurer ABC's total fair value loss reserve as of that date is \$189.19. Risk margins and fair value reserves can also be easily determined for the individual reserve cells corresponding to accident years 2007 through 2009.[23]

ABC's *solvency capital requirement* $C^{R}_{ABC\,,\,2009}$ as of Dec. 31, 2009 can be determined using Formula (3.11), rewritten below with appropriate subscripts:

$$C^{R}_{ABC\,,\,2009} = \frac{F_{ABC\,,\,2010} + f_{ABC\,,\,2010} \cdot R'_{ABC\,,\,2010}}{1 + roe_{PT\,,\,0}} \tag{4.2}$$

In Formula (4.2), $F_{ABC\,,\,2010}$ refers to the amount required as of Dec. 31, 2010 to take the present value loss funding of ABC's *own* claim obligations up to the 99.5 percent confidence level. Let's assume the 99.5 percent confidence level for ABC's Line B loss development to be about 2.74 standard deviations above the mean (slightly higher than the 2.71 standard deviations we observed for Lines A, B and C combined at the industry level). Insurer ABC's 15 percent greater reserve volatility implies that its standard deviation is about 9.3 percent (115 percent of the industry's 8.1 percent) of the expected hindsight estimate of \$185.29 (10% of $\$1{,}793 \times 1.0334$). Then Formula (3.2) implies $F_{ABC\,,\,2010} = NSD_{ABC\,,\,2010}(99.5\%) \cdot \sigma_{ABC\,,\,2010} = \$47.42$.[24]

$f_{ABC\,,2010}$ is the fraction by which insurer ABC's Dec. 31, 2010 *own* present value unpaid losses embedded in the one-year hindsight estimate at the 99.5 percent confidence level exceed the expected present value unpaid loss amount at that time. We assume $f_{ABC\,,2010} = 0.32$, which is higher than the comparable figure at the industry all-lines total level and reflects the greater variability and skewness associated with insurer ABC's smaller and less diversified portfolio.

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[23] We leave that exercise for the reader.

[24] Slight differences arise from these reported numbers if the calculations are performed with rounded inputs.

$R'_{ABC,\,2010}$ is the expected risk margin as of Dec. 31, 2010. Its value can be determined using Formula (3.12) or that formula's less intimidating variation below:

$$\mathrm{R}'_{ABC,\,2010} = coeff_{2010,\,1} \cdot F^{T}_{ABC,\,2011} + coeff_{2010,\,2} \cdot F^{T}_{ABC,\,2012} \tag{4.3}$$

where

$$coeff_{2010,\,1} = v_{1:1} \cdot \frac{roe_{PT,\,1} - r_{1:1}}{1 + roe_{PT,\,1}} \tag{4.4}$$

and

$$coeff_{2010,\,2} = v_{1:1} \cdot v_{2:1} \cdot (f_{T,\,n+2} \cdot \frac{roe_{PT,\,1} - r_{1:1}}{1 + roe_{PT,\,1}} + 1) \cdot \frac{roe_{PT,\,2} - r_{2:1}}{1 + roe_{PT,\,2}} \tag{4.5}$$

Using the risk-free yield curve, return on equity and $f_{T,\,2011}$ assumptions described earlier in this section, the values of $coeff_{2010,\,1}$ and $coeff_{2010,\,2}$ are 0.1055 and 0.1054, respectively. Given our assumption that insurer ABC carries 10 percent of Line B reserves but with one-year development volatility 15 percent greater than the industry, the values of $F^{T}_{ABC,\,2011}$ and $F^{T}_{ABC,\,2012}$ can be calculated as $0.1 \times \$295 \times 1.15 = \$33.92$ and $0.1 \times \$174 \times 1.15 = \$20.04$, respectively, where \$295 and \$174 are the corresponding industry-level *F* values for Line B found in Tables B-2 and B-3. Using those values in Formula (4.3), we obtain a year-end 2010 risk margin of $R'_{ABC,\,2010} = 0.1055 \times \$33.92 + 0.1054 \times \$20.04 = \$5.69$.

Given the values we have just computed for $F_{ABC,\,2010}$, $f_{ABC,2010}$ and $R'_{ABC,\,2010}$, Formula (4.2) yields required *solvency capital* for insurer ABC as of Dec. 31, 2009 of:

$$C^{R}_{ABC,\,2009} = \frac{\$47.42 + 0.32 \cdot \$5.69}{1.1584} = \$42.51.$$

In contrast, the *market-clearing capital* $C^{T}_{ABC,\,2009}$ attributable to ABC's loss reserve portfolio as of Dec. 31, 2009 is given by Formula (3.10), rewritten as follows:

$$C^{T}_{ABC,\,2009} = \frac{F^{T}_{ABC,\,2010} + f_{T,\,2010} \cdot \mathrm{R}'_{ABC,\,2010}}{1 + roe_{PT,\,0}} \tag{4.6}$$

$$= \frac{\$40.93 + 0.22 \cdot \$5.69}{1.1584} = \$36.41$$

where $F^{T}_{ABC,\,2010} = 0.1 \times F^{T}_{C,\,2010} \times 1.15 = \$40.93$.

The 2010 cost of the market-clearing capital supporting ABC's Dec. 31, 2009 fair value loss reserve of \$189.19 is $(roe_{PT,\,0} - r_1) \cdot C^{T}_{ABC,\,2009} = 0.125 \times \$36.41 = \$4.55$. That cost is funded by the partial amortization of the risk margin $R'_{ABC,\,2009}$ during 2010. After accumulating interest at rate $r_1 = 3.34\%$ through Dec. 31, 2010, the initial risk margin $R'_{ABC\,,\,2009}$ in insurer ABC's fair value loss reserves of \$9.91 is expected to be replaced by the updated risk margin $R'_{ABC\,,\,2010} = \$5.69$, which implies expected risk margin amortization of $R'_{ABC\,,\,2009} \cdot (1 + r_1) - R'_{ABC\,,\,2010} = \$9.91 \times 1.0334 - \$5.69$, or \$4.55.

Because ABC's required solvency capital is greater than the market-clearing capital, the risk margin amortization of \$4.55 produces a lower expected rate of return on equity with respect to required solvency capital. While the expected total pretax return on market-clearing capital during 2010 is 15.84 percent ($12.5\% + 3.34\%$), the comparable return on required solvency capital is only 14.04 percent ($\$4.55 / \$42.51 + 3.34\%$).

In this section we have illustrated the calculation of market-clearing risk margins and resulting fair value reserves by hypothetical industry line of business and accident year reserve cell as well as in total for a hypothetical insurer ABC. Those risk margins and fair value reserves are additive, i.e., the sum of reserve cell amounts matches the result obtained by applying the formulas at the subtotal or total level.

We also illustrated the calculation of required solvency capital for hypothetical monoline insurer ABC and contrasted it with the implied market-clearing capital underpinning the market-consistent risk margin in ABC's fair value loss reserves. In our illustration, the volatility of ABC's own loss development was such that the required solvency capital exceeded the market-clearing capital by a substantial amount. As a consequence, insurer ABC's expected return on equity during 2010 with respect to required solvency capital was lower (14.04 percent) than the 15.84 percent contemplated by the market-clearing risk margin. Note the similarity of this situation with the Capital Asset Pricing Model, which predicts that investors can expect to be compensated only to the extent that its risks are residual undiversifiable market risks. Under these circumstances, insurer ABC has an incentive to reduce its own required solvency capital by minimizing its exposure to reducible loss reserve development volatility. That could be achieved in a number of ways, ranging from optimizing its loss reserving process to eliminate variability preventable by use of better techniques and/or better data, to entering additional lines of business to increase diversification, to transferring some or all of its existing loss reserves to a third party at a price equal to their fair value.

5. Summary and Conclusions

In this paper we have described a relatively minor modification to the Wacek [4] framework for the determination of fair value loss reserves with embedded market-consistent cost-of-capital risk margins. Our modification addresses and resolves two problems with that framework in its original form, namely, that its risk margins are not additive and they are based on a capital requirement that is not market-consistent, despite that being the stated intent. These problems are also inherent in the Solvency II specifications for its putatively market-consistent risk margins on which Wacek based his framework. The implication is that Solvency II risk margins, as defined in the final CEIOPS (October 2009) advice for Level 2 implementing measures [1], are not market-consistent.[25]

We addressed the market-consistency problem by shifting the capital focus away from solvency capital at the insurer level and toward the hypothetical solvency capital requirement for the insurance industry as a whole. Then, in the context of a value-at-risk-based capital adequacy standard and a competitive market, we showed how to disaggregate that total market capital into additive components reflecting the exposure contribution of each risk source to the total. We argued that, in a competitive market, the capital attributed to a loss reserve portfolio, especially in the context of a transfer to a third party, can be expected to match the contribution of that portfolio to the total industry reserve development risk.

Our key innovation was to replace Wacek's F_{n+i+1}, which is based solely on the internal characteristics of a loss portfolio, with $F_{T,\,n+i+1} = \sum_{j}^{m} F_{j,\,n+i+1}^{T}$, where $F_{j,\,n+i+1}^{T}$ is an additive measure based on a combination of a loss portfolio's internal characteristics *and* its correlation with the characteristics of the total industry portfolio:

$$F_{j,\,n+i+1}^{T} = NSD_{T,\,n+i+1}(\alpha) \cdot \rho_{j,\,n+i+1} \cdot \sigma_{j,\,n+i+1} \tag{3.5}$$

An obstacle to the adoption of our modification to the Wacek framework is the difficulty in determining the correlation coefficient of claim source *j* loss development with the loss development of the industry as a whole. However, there are now multiple sources of U.S. Annual Statement Schedule P data that could be tapped for an analysis of total industry development characteristics.[26] One valuable avenue for future research and development would be the analysis, compilation and publication of U.S. (or, even better, global) industry loss development volatility in a form that could then be used by individual insurers to determine correlation coefficients for their internal reserve cells.

Getting the fair value question right is important because it is so intertwined with economic capital. The amount of *available* economic capital, for example, depends on the fair value adjustment to full value loss reserves, while *required* economic capital reflects the volatility of fair value loss reserves. We will not get economic capital right unless we also get fair value loss reserves right, a goal toward which we hope we have made a small contribution.

[25] CEIOPS is the acronym for the Committee of European Insurance and Occupational Pensions Supervisors.

[26] A.M. Best, Highline and SNL are three such sources. There may be others.

Appendix A

Derivation of Formula (2.3)

Given any sum of random variables $x_T = \sum_j^m x_j$, the total variance σ_T^2 can be expressed as the sum of covariances:

$$\sigma_T^2 = \sigma_{1T} + \sigma_{2T} + \sigma_{3T} + \cdots + \sigma_{mT} \tag{A.1}$$

where $\sigma_{1T}, \sigma_{2T}, \sigma_{3T}, \ldots, \sigma_{mT}$ represent the respective covariances between component random variables $x_1, x_2, x_3, \ldots, x_m$ and the total x_T.

The coefficient of correlation between component random variable x_j and the total x_T for $1 \le j \le m$ is given by:

$$\rho_{jT} = \frac{\sigma_{jT}}{\sigma_j \cdot \sigma_T}. \tag{A.2}$$

Using the implication from Formula (A.2) that $\sigma_{iT} = \rho_{iT} \cdot \sigma_i \cdot \sigma_T$, Formula (A.1) can be rewritten as:

$$\sigma_T^2 = \sigma_T \cdot (\rho_1 \cdot \sigma_1 + \rho_2 \cdot \sigma_2 + \rho_3 \cdot \sigma_3 + \cdots + \rho_m \cdot \sigma_m). \tag{A.3}$$

The σ_T on both sides of Formula (A.3) can be eliminated, which results in the following Formula (A.4), which expresses the total standard deviation σ_T as the weighted sum of the component standard deviations with the correlation coefficients as weights:

$$\sigma_T = \rho_{1T} \cdot \sigma_1 + \rho_{2T} \cdot \sigma_2 + \rho_{3T} \cdot \sigma_3 + \cdots + \rho_{mT} \cdot \sigma_m \tag{A.4}$$

Appendix B

Derivation of Direct Formulas for $\boldsymbol{R'_n}$ and $\boldsymbol{R'_{n+1}}$

It is possible to derive a direct formula for R'_{n+i} for $0 \le i \le k-1$ by recursively expanding Formula (2.7W). First we eliminate C^{R}_{n+i} by substituting $C^{\mathrm{R}}_{n+i} = \dfrac{F_{n+i+1} + f_{n+i+1} \cdot \mathrm{R}'_{n+i+1}}{1+roe_{PT}}$ from Formula (2.23W):

$$\mathrm{R}'_{n+i} = v_{i:1} \cdot ((roe_{PT} - r_{i:1}) \cdot C^{\mathrm{R}}_{n+i} + \mathrm{R}'_{n+i+1}), \qquad (2.7W)$$

$$= v_{i:1} \cdot ((roe_{PT} - r_{i:1}) \cdot \frac{F_{n+i+1} + f_{n+i+1} \cdot \mathrm{R}'_{n+i+1}}{1+roe_{PT}} + \mathrm{R}'_{n+i+1})$$

$$= v_{i:1} \cdot \frac{roe_{PT} - r_{i:1}}{1+roe_{PT}} \cdot F_{n+i+1} + v_{i:1} \cdot (f_{n+i+1} \cdot \frac{roe_{PT} - r_{i:1}}{1+roe_{PT}} + 1) \cdot \mathrm{R}'_{n+i+1} \qquad (B.1)$$

Using Formula (B.1) with $i=1$, we obtain Formulas (B.2) for R'_{n+1}:

$$\begin{aligned} \mathrm{R}'_{n+1} &= v_{1:1} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} \cdot F_{n+2} \\ &+ v_{1:1} \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot R'_{n+2} \end{aligned} \qquad (B.2)$$

Substituting the expression for R'_{n+2} given by Formula (B.1) into Formula (B.2):

$$\begin{aligned} R'_{n+1} &= v_{1:1} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} \cdot F_{n+2} \\ &+ v_{1:1} \cdot v_{2:1} \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{2:1}}{1+roe_{PT}} \cdot F_{n+3} \\ &+ v_{1:1} \cdot v_{2:1} \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot (f_{n+3} \cdot \frac{roe_{PT} - r_{2:1}}{1+roe_{PT}} + 1) \cdot R'_{n+3} \end{aligned} \qquad (B.3)$$

By successive substitutions of $R'_{n+3} \ldots R'_{n+k-1}$ with the expressions implied by Formula (B.1) we obtain the following fully expanded formula for R'_{n+1}:

$$\begin{aligned} R'_{n+1} &= v_{1:1} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} \cdot F_{n+2} \\ &+ v_{1:1} \cdot v_{2:1} \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{2:1}}{1+roe_{PT}} \cdot F_{n+3} \\ &+ \cdots \\ &+ v_{1:1} \cdot v_{2:1} \cdots v_{k-1:1} \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdots (f_{n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{k-1:1}}{1+roe_{PT}} \cdot F_{n+k} \end{aligned} \qquad (B.4)$$

where the final term involving R'_{n+k} has dropped out because $R'_{n+k} = 0$.

Formula (B.1) with $i=0$ yields the following formula for R'_n:

$$R'_n = v_1 \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} \cdot F_{n+1} + v_1 \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1 + roe_{PT}} + 1) \cdot R'_{n+1}. \qquad \text{(B.5)}$$

Substituting the expression for R'_{n+1} into Formula (B.5) and rearranging terms, we obtain Formula (3.1) from the body of the paper:

$$\begin{aligned} R'_n &= v_1 \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} \cdot F_{n+1} \\ &+ v_2^2 \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} \cdot F_{n+2} \\ &+ v_3^3 \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{2:1}}{1+roe_{PT}} \cdot F_{n+3} \qquad (3.1) \\ &+ \cdots \\ &+ v_k^k \cdot (f_{n+1} \cdot \frac{roe_{PT} - r_1}{1+roe_{PT}} + 1) \cdot (f_{n+2} \cdot \frac{roe_{PT} - r_{1:1}}{1+roe_{PT}} + 1) \cdots (f_{n+k-1} \cdot \frac{roe_{PT} - r_{k-2:1}}{1+roe_{PT}} + 1) \cdot \frac{roe_{PT} - r_{k-1:1}}{1+roe_{PT}} \cdot F_{n+k} \end{aligned}$$

where, for *m* an integer, $v_m^m = v_1 \cdot v_{1:1} \cdot v_{2:1} \cdots v_{m-1:1}$.

Abbreviations and Notations

α	=	confidence level (probability) that insolvency can be avoided
$coeff_{n,\,i+1}$	=	coefficient for $F^{T}_{j,\,n+i+1}$ in Formula (4.2)
C^{R}_{n}	=	required *solvency* capital at time *n*; sometimes denoted $C^{R}_{j,\,n}$ or $C^{R}_{T,\,n}$
C^{R}_{T}	=	required *solvency* capital for risk *T* (basic definition ignoring time)
$C^{T}_{j,\,n}$	=	market-clearing capital for risk *j* at time *n*
$E(\cdot)$	=	expected value operator
F_{n+1}	=	additional amount of assets needed at time $n+1$ to bring present value loss funding up to the α confidence level; sometimes denoted $F_{j,\,n+1}$
$F^{T}_{j,\,n+1}$	=	contribution of risk *j* to $F_{T,\,n+1}$ (the additional amount of assets needed at time $n+1$ to bring total industry present value loss funding up to the α confidence level)
f_{n+1}	=	fraction by which the time *n*+1 unpaid losses embedded in the one-year hindsight estimate at α confidence level exceeds the expected time *n*+1 one-year hindsight estimate; sometimes denoted $f_{j,\,n+1}$ or $f_{T,\,n+1}$
h_{n+1}	=	$l_{n+1} + p_{n+1}$ = random variable, at time *n*, for one-year hindsight losses as of time *n*+1, given L_n
i	=	integer subscript denoting a number of years beyond the initial valuation date at time *n*, $0 \le i \le k-1$
j	=	integer subscript denoting a risk source, $0 \le j \le m$
k	=	integer number of years of loss payments beyond time *n*
L_n	=	unpaid losses at time *n*
l_{n+1}	=	random variable, at time *n*, for unpaid losses as of time *n*+1, given L_n
m	=	integer denoting number of components comprising a total
n	=	integer subscript denoting the first of a sequence of annual loss reserve valuation dates (time *n*+*i* is *i* years later)
$NSD(\alpha)$	=	Number of standard deviations above the mean corresponding to α
p_{n+1}	=	random variable, at time *n*, for paid losses between time *n* and *n*+1, given L_n

$PV(\cdot)$	=	risk-free present value operator
$PV(h_{n+1})$	=	$PV(L_{n+1}) + p_{n+1} \cdot (1+\frac{1}{2}r)$ = random variable, at time *n*, for the present value of h_{n+1} as of time *n*+1, given L_n
$PV(L_{n+1} \mid VaR_{\alpha}(PV(h_{n+1}))$	=	present value of the unpaid loss component of the one-year hindsight loss estimate a the α confidence level
R'_n	=	risk-free present value of future risk charges associated with unpaid losses L_n at time *n*; sometimes denoted $R'_{j,n}$
$r_{f:m}$	=	risk-free annual *f*-year forward interest rate on the *m*-year maturity bond for the period from time *n*+*f* to *n*+*f*+*m*
r_m	=	risk-free annual interest rate for the *m*-year maturity bond for the period from time *n* to *n*+*m*
roe_{PT}	=	annualized required pretax return on equity (capital)
$roe_{PT,i}$	=	annualized required pretax return on equity (capital) in *i*-th year after *n*
σ_j	=	standard deviation of risk *j*
σ_{jT}	=	covariance of risk *j* with total risk *T*, of which *j* is a component
ρ_{jT}	=	correlation coefficient of risk *j* with total risk *T*, of which *j* is a component
T	=	a risk source, typically a Total comprising multiple components
$T(L_n)$	=	fair value at time *n* of unpaid losses L_n
t_{n+1}	=	$T(l_{n+1} + p_{n+1})$ = $T(l_{n+1}) + p_{n+1} \cdot (1+\frac{1}{2}r)$ = random variable, at time *n*, for fair value at time *n*+1 of one-year hindsight estimate of L_n
v	=	$(1+r)^{-1}$ = one-year risk-free discount factor assuming a flat yield curve
$v_{f:m}$	=	$(1+r_{f:m})^{-1}$ = one-year risk-free discount factor corresponding to $r_{f:m}$
v_m	=	$(1+r_m)^{-1}$ = one-year risk-free discount factor r_m
$VaR_{\alpha}(t_{n+1})$	=	value-at-risk with respect to t_{n+1} at the α confidence level
$VaR_{\alpha}(T)$	=	value-at-risk with respect to risk *T* at the α confidence level
$VaR_{\alpha}(PV(h_{n+1}))$	=	value-at-risk with respect to $PV(h_{n+1})$ at the α confidence level
x_j	=	random variable for risk *j*

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Michael Wacek is executive vice president and chief risk officer of Odyssey Re Holdings Corp., based in Stamford, Conn. Over the course of more than 30 years in the property-casualty insurance industry, including nine years in the London market, Mike has seen the business from the vantage point of a primary insurer, reinsurance broker and reinsurer in actuarial, underwriting and executive management roles. He has a Bachelor of Arts degree in math and economics from Macalester College, is a fellow of the Casualty Actuarial Society and a member of the American Academy of Actuaries. He has authored a number of papers on risk and insurance issues.