Computing Tight Bounds for Insurance Payments with Nonlinear Risk

Man Hong WONG[1] Shuzhong ZHANG[2]

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(SDP) \quad \inf \quad & C \bullet X \\
\text{s.t. } & A_i \bullet X \leq b_i \quad \forall i = 1, \cdots, n \\
& X \succeq 0
\end{aligned}
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where $A \bullet B := tr(A^T B)$

- whole matrix X is a variable
- $X \succeq 0$ means X is a semidefinite matrix (all eigenvalues of X are nonnegative)
- applications in engineering and finance
- any problem arriving at this form can be solved efficiently!

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Motivation

$$? \le \mathbb{E}[\psi(x)] \le ?$$

- when distribution is not known
- difficult to estimate the distribution, e.g. extreme events
- only some realizations of x exist $\rightarrow$ moments can be estimated
- efficiently find the numerical bounds?
- $$\sup_{x \sim (m_1, \cdots, m_n)} \mathbb{E}[\psi(x)]$$

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Brief Review

- analytical form: $\psi(x)$ is (piecewise) linear: Scarf (1958), Jansen et al (1986), Lo (1987), Cox (1991)
- numerical ways with semidefinite programming (SDP): Bertsimas & Popescu (2000), Popescu (2005), Cox et al (2008), He et al (2010)
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An example on mortgage payment

Recall

$$P = A\left(\frac{1}{1+r} + \cdots + \frac{1}{(1+r)^t}\right) = A\frac{(1+r)^t - 1}{r(1+r)^t}$$

$$f_{P,t}(r) := A = \frac{Pr(1+r)^t}{(1+r)^t - 1}$$

- How worst can $\mathbb{E}(f_{P,t}(r))$ be? $\to \sup \mathbb{E}[f_{P,t}(r)]$?
- bound for stop-loss insurance? $\to \sup \mathbb{E}[(f_{P,t}(r) - h)_+]$
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Experiential Scenario

- loan $1000, 20 periodic payments in return
- floating rate (assume latest 2.5%, so $f_{1000,20}(0.025) = \$51.32$.)
- 12-month Hong Kong Dollar Interest Rate (take 5 years, 10 years and 20 years samples)

period	μ	σ	$\sup \mathbb{E}[f_{1000,20}(r)]$	$\frac{\sup \mathbb{E}[f_{1000,20}(r)]}{f_{1000,20}(0.025)} - 1$
5-year	1.45%	1.25%	$58.2117	13%
10-year	1.27%	1.21%	$57.0003	11%
20-year	3.60%	2.50%	$71.9524	40%

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period	$\mu + \sigma$	eqv. h[1]	$\sup \mathbb{E}[f_{1000,20}(r) - h]_+$	$\sup \mathbb{P}(f_{1000,20}(r) \geq h)$
5-year	2.09%	$61.6892	$2.3786	0.6938
10-year	1.93%	$60.7444	$2.3082	0.6580
20-year	4.07%	$74.0386	$7.1618	0.8845

period	$\mu + 2\sigma$	eqv. h[2]	$\sup \mathbb{E}[f_{1000,20}(r) - h]_+$	$\sup \mathbb{P}(f_{1000,20}(r) \geq h)$
5-year	3.23%	$68.6531	$1.3078	0.3303
10-year	3.05%	$67.5268	$1.2222	0.3161
20-year	5.91%	$86.5486	$4.1012	0.5394

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- interest rate (in a broad sense)
- mortgage payments
  $\rightarrow x$ is mortgage rate
- annuity life insurance
  $\rightarrow x$ is discounted rate
- bond options
  $\rightarrow x$ is bond yield
- ... may be more!

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Q&A

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