On the Haezendonck-Goovaerts Risk Measure for Extreme Risks [1]

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[1] Based on a joint work with Qihe Tang

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Definitions

Let X be a real-valued random variable, representing a risk variable in loss-profit style, with a distribution function F on $\mathbb{R}$.

A function $\varphi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is called a normalized Young function if it is continuous and strictly increasing with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$.

For $q \in (0, 1)$, the Haezendonck-Goovaerts risk measure for X is defined as

$$H_q[X] = \inf_{x \in \mathbb{R}} \left(x + H_q[X, x] \right), \tag{1}$$

where $H_q[X, x]$ is the unique solution of the equation

$$\mathbb{E}\left[\varphi \left(\frac{(X - x)_+}{H_q[X, x]} \right) \right] = 1 - q \tag{2}$$

if $\overline{F}(x) > 0$ and let $H_q[X, x] = 0$ otherwise.

A short literature review

- First introduced by Haezendonck and Goovaerts (1982)
- Named as the Haezendonck risk measure by Goovaerts, Kaas, Dhaene and Tang (2004)
- We think that it is more proper to call it the Haezendonck-Goovaerts risk measure.
- Recently studied by Bellini and Rosazza Gianin (2008a, 2008b) and Krätschmer and Zähle (2011).
- Usually, the Young function $\varphi(\cdot)$ is assumed to be convex so that the Haezendonck-Goovaerts risk measure $H_q[X]$ is a law invariant coherent risk measure.

A special case

The special case is $\varphi(t) = t$ for $t \in \mathbb{R}_+$. Then

$$H_q[X] = \inf_{x \in \mathbb{R}} \left(x + \frac{\mathbb{E}\left[(X - x)_+\right]}{1 - q} \right) = \frac{1}{1 - q} \int_q^1 F^{\leftarrow}(p) \mathrm{d}p$$

and, thus, the Haezendonck-Goovaerts risk measure is reduced to the well-known Conditional Tail Expectation risk measure.

For a proper distribution function F and for $p \in [0, 1]$,

$$F^{\leftarrow}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$$

denotes the inverse function of F, also called the quantile of F or the Value at Risk of X at level p.

Remarks

The parameter q in the definition of the Haezendonck-Goovaerts risk measure vaguely represents the confidence/risk aversion level.

We shall focus on the asymptotic behavior of $H_q[X]$ as $q \uparrow 1$.

Let $\hat{x} = \sup\{x \in \mathbb{R} : F(x) < 1\} \leq \infty$ be the upper endpoint of X and $\hat{p} = \Pr(X = \hat{x})$. We only consider $\hat{p} = 0$. In this case,

$$\lim_{q \uparrow 1} H_q[X] = \hat{x}.$$

- When $\hat{x} = \infty$ we shall establish exact asymptotics for $H_q[X]$ diverging to ∞ as $q \uparrow 1$;
- When $\hat{x} < \infty$ we shall establish exact asymptotics for $\hat{x} - H_q[X]$ decaying to 0 as $q \uparrow 1$.

A power Young function

Due to the complexity of the problem, we shall only consider a power Young function

$$\varphi(t) = t^k, \qquad k \geq 1.$$

This ensures the convexity of the Young function $\varphi(\cdot)$ and, hence, the coherence of the Haezendonck-Goovaerts risk measure.

Since $H_q[X] = \mathrm{CTE}_q[X]$ when $k = 1$ while $\mathrm{CTE}_q[X]$ has been extensively investigated, we shall consider $k > 1$ only.

Definition and Fisher-Tippett theorem

A distribution function F on $\mathbb{R}$ is said to belong to the max-domain of attraction of an extreme value distribution function G if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| F^n(c_n x + d_n) - G(x) \right| = 0$$

holds for some norming constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \in \mathbb{N}$.

By the classical Fisher-Tippett theorem (see Fisher and Tippett (1928) and Gnedenko (1943)), only three choices for G are possible, namely the Fréchet, Weibull and Gumbel distributions.

Three cases

The Fréchet distribution function is given by $\Phi_\gamma(x) = \exp\{-x^{-\gamma}\}$ for $x > 0$. A distribution function F belongs to $\mathrm{MDA}(\Phi_\gamma)$ if and only if

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\gamma}, \qquad y > 0.$$

A typical example is Pareto distribution.

The Weibull distribution function is given by $\Psi_\gamma(x) = \exp\{-|x|^\gamma\}$ for $x \le 0$. A distribution function F belongs to $\mathrm{MDA}(\Psi_\gamma)$ if and only if $\hat{x} < \infty$ and

$$\lim_{x\downarrow 0} \frac{\overline{F}(\hat{x} - xy)}{\overline{F}(\hat{x} - x)} = y^{\gamma}, \qquad y > 0.$$

Almost all continuous distributions with bounded supports belong to $\mathrm{MDA}(\Psi_\gamma)$.

Three cases (Cont.)

The standard Gumbel distribution function is given by $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$ for $x \in \mathbb{R}$. A distribution function F with a right endpoint $\hat{x}$ belongs to $\mathrm{MDA}(\Lambda)$ if and only if

$$\lim_{x \uparrow \hat{x}} \frac{\overline{F}(x + ya(x))}{\overline{F}(x)} = \mathrm{e}^{-y}, \qquad y \in \mathbb{R},$$

for some auxiliary function $a(\cdot) : (-\infty, \hat{x}) \longmapsto \mathbb{R}_+$. A commonly-used choice of $a(\cdot)$ is the mean excess function,

$$a(x) = \mathbb{E}\left[X - x | X > x\right] \quad \text{for } x < \hat{x}.$$

Almost all rapidly varying distributions belong to $\mathrm{MDA}(\Lambda)$.

Main result for the Fréchet case

Theorem 1. Let $\varphi(t) = t^k$ for $t \geq 0$ for some $k > 1$ and let $F \in \mathrm{MDA}(\Phi_\gamma)$ for some $\gamma > k > 1$. Then, as $q \uparrow 1$,

$$H_q[X] \sim \frac{\gamma(\gamma - k)^{k/\gamma - 1}}{k^{(k-1)/\gamma}} \left(B(\gamma - k, k)\right)^{1/\gamma} F^{\leftarrow}(q). \qquad (3)$$

Numerical results

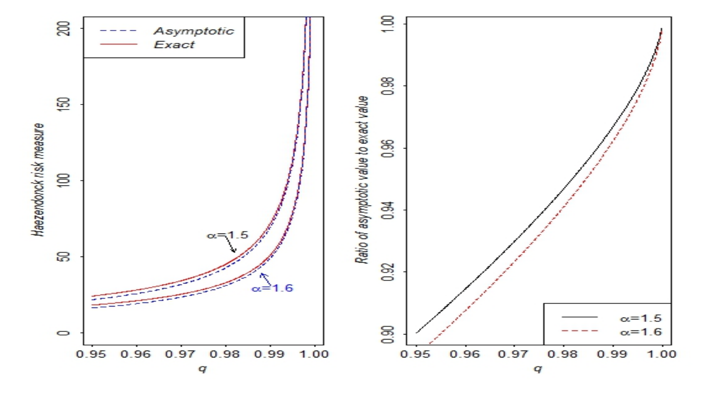
Assume that F is the Pareto distribution with parameters $\alpha > 0$ and $\theta > 0$:

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \qquad x \in \mathbb{R}_+.$$

We numerically compute the exact value of $H_q[X]$. We compute the asymptotic value of $H_q[X]$ according to Theorem 1.

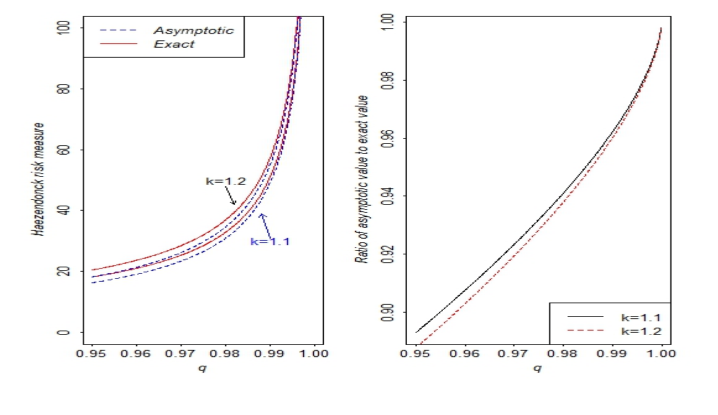
Graph 1

Graph 1. $\alpha = 1.5$ and 1.6, $k = 1.1$ and $\theta = 1$.



Graph 2

Graph 2. $k = 1.1$ and 1.2, $\alpha = 1.6$ and $\theta = 1$.



Main result for the Weibull case

Theorem 2. Let $\varphi(t) = t^k$ for $t \geq 0$ for some $k > 1$ and let $F \in \mathrm{MDA}(\Psi_\gamma)$ with $\gamma > 0$ and $0 < \hat{x} < \infty$. Then, as $q \uparrow 1$,

$$\hat{x} - H_q[X] \sim \frac{\gamma}{\gamma + k} \left(\frac{k^{k-1}}{B(\gamma + 1, k)(\gamma + k)^k} \right)^{1/\gamma} (\hat{x} - F^{\leftarrow}(q)).$$

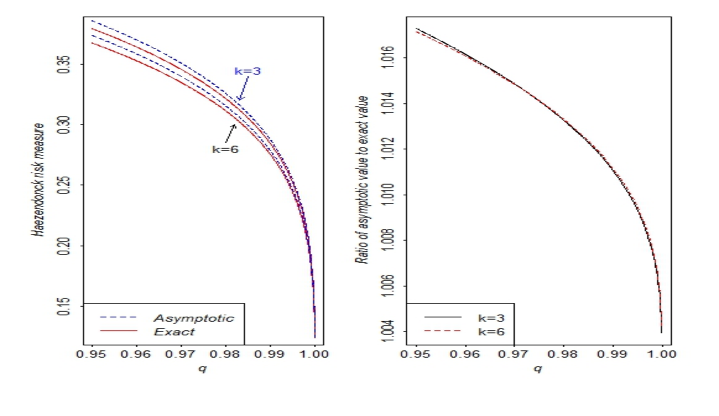
Numerical results

Assume that F is the Beta distribution with parameters $a > 0$ and $b > 0$:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \qquad 0 < x < 1.$$

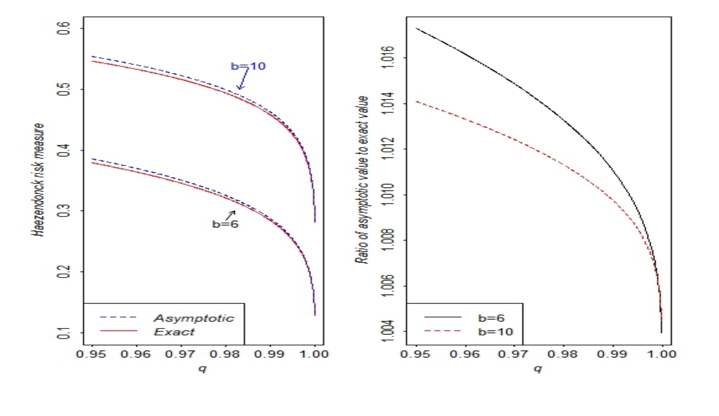
Graph 3

Graph 3. $k = 3$ and 6, $a = 2$ and $b = 6$



Graph 4

Graph 4. $b = 6$ and 10, $k = 3$ and $a = 2$



Main result for the Gumbel case

Theorem 3. Let $\varphi(t) = t^k$ for $t \geq 0$ for some $k > 1$ and let $F \in \mathrm{MDA}(\Lambda)$ with an auxiliary function $a(\cdot)$ and an upper endpoint $0 < \hat{x} \leq \infty$. Then, as $q \uparrow 1$,
(i) when $\hat{x} = \infty$ we have

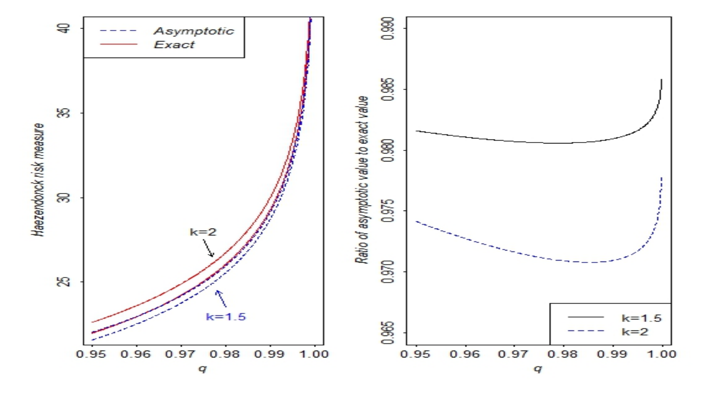
$$H_q[X] \sim F^{\leftarrow}\left(1 - \frac{k^{k-1}}{\Gamma(k)}(1-q)\right);$$

(ii) when $\hat{x} < \infty$ we have

$$\hat{x} - H_q[X] \sim \hat{x} - F^{\leftarrow}\left(1 - \frac{k^{k-1}}{\Gamma(k)}(1-q)\right).$$

Numerical result

Graph 5. F = Lognormal ($\mu = 2, \sigma = 0.5$), $k = 1.5$ and 2.



Conclusion and future work

We have done the following:

- for the Fréchet case, $H_q[X] \sim c_1 F^{\leftarrow}(q)$
- for the Weibull case, $\hat{x} - H_q[X] \sim c_2 (\hat{x} - F^{\leftarrow}(q))$
- for the Gumbel case,

$$\begin{cases} H_q[X] \sim F^{\leftarrow}(1 - c_3 q), & \text{when } \hat{x} = \infty, \\ \hat{x} - H_q[X] \sim (\hat{x} - F^{\leftarrow}(1 - c_3 q)), & \text{when } \hat{x} < \infty. \end{cases}$$

Future work:

- Extend to a general Young function $\varphi(\cdot)$;
- Derive second-order asymptotics to improve the accuracy.

Thank you!