Actuarial Present Values Accounting for Common Shock

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Abstract: We study the problem of how a common disaster or life lengthening force affects two independent lives. We specifically look at two exponential lives and determine different actuarial present values for the effect of a common disaster, life lengthening force, and the effects of both the common disaster and life lengthening.

Common Disaster

In this section we determine the probability of two individuals terminating due to either biological causes specific to each life or a common disaster that affects both, whichever occurs first. Therefore, we can separate causes of termination into both biological causes which are independent from each other and a common disaster.

The probability that the biological component terminates the individual during $(k, k+1]$ is $(1-q_i)^k q_i = p_i^k q_i$, where $p_i = e^{u_i}, q_i = 1 - p_i$, and $k = 0, 1, 2$. The probability that life i survives one year through biological causes is denoted by p_i. Define K_i as the number of years individual i lives without terminating due to biological causes.

Let's say that a disaster occurs at time D. This disaster is independent of life i where $i = 1, 2...$ We will assume that individual lives are independent from each other and from the disaster D. The disaster and lives will be modeled by geometric variables. The probability that the catastrophe occurs during $(k, k+1]$ is $(1-d)^k d, k = 0, 1, 2, ...,$ where d is between zero and one. The larger d is, the sooner the disaster is expected to occur. d is the chance the disaster will occur in any one year while $(1-d)$ is the chance that there is no disaster in that year. To determine the probability that the disaster occurs after time n, we sum up the probability from time n+1 to infinity.

$$\begin{aligned} P(D > n) &= \sum_{k=n+1}^{\infty} (1-d)^k d \\ &= d\frac{(1-d)^{n+1}}{1-(1-d)} \\ &= (1-d)^{n+1}. \end{aligned}$$

$K_i^D = min(K_i, D)$,where K_i is defined as the the whole number of years lived by an individual i without a disaster. The lives K_1^D and K_2^D are dependent through the common disaster. In absence of the common disaster, K_1 and K_2 are independent.

To calculate $P(K_1^D = n_1, K_2^D = n_2)$ we look at

$$\begin{aligned} P(K_1^D > n_1, K_2^D > n_2) &= P(D > n_1, D > n_2, K_1 > n_1, K_2 > n_2) \\ &= P(D > max(n_1, n_2)) * P(K_1 > n_1) * P(K_2 > n_2) \\ &= (1-d)^{max(n_1,n_2)+1} p_1^{n_1+1} p_2^{n_2+1}. \end{aligned} \tag{1}$$

We then use the above formula to look at the tail probabilities to determine the probability mass function denoted as $P(K_1^D = n_1, K_2^D = n_2)$. This is the same as looking at the box from $n_1 - 1, n_2 - 1, n_1$ and n_2 and subtracting off sections to determine $P(K_1^D = n_1, K_2^D = n_2)$.

For $n_1, n_2 = 0, 1, 2...$

$$
\begin{aligned}
P(K_1^D = n_1, K_2^D = n_2) = {} & P(K_1^D > n_1 - 1, K_2^D > n_2 - 1) \\
& - P(K_1^D > n_1, K_2^D > n_2 - 1) \\
& - P(K_1^D > n_1 - 1, K_2^D > n_2) \\
& + P(K_1^D > n_1, K_2^D > n_2).
\end{aligned}
$$

This is the same as

$$
\begin{aligned}
P(K_1^D = n_1, K_2^D = n_2) = {} & (1-d)^{max(n_1-1, n_2-1)+1} p_1^{(n_1-1)+1} p_2^{(n_2-1)+1} \\
& - (1-d)^{max(n_1, n_2-1)+1} p_1^{n_1+1} p_2^{(n_2-1)+1} \\
& - (1-d)^{max(n_1-1, n_2)+1} p_1^{(n_1-1)+1} p_2^{n_2+1} \\
& + (1-d)^{max(n_1, n_2)+1} p_1^{n_1+1} p_2^{n_2+1} \\
\\
= {} & (1-d)^{max(n_1, n_2)} p_1^{n_1} p_2^{n_2} \\
& - (1-d)^{max(n_1+1, n_2)} p_1^{n_1+1} p_2^{n_2} \\
& - (1-d)^{max(n_1, n_2+1)} p_1^{n_1} p_2^{n_2+1} \\
& + (1-d)^{max(n_1+1, n_2+1)} p_1^{n_1+1} p_2^{n_2+1}.
\end{aligned}
$$

We then look at specific cases for different values of n_1, n_2. For $n_1 = 0, 1, ..., n_2 - 1$, we have

$$
\begin{aligned}
P(K_1^D = n_1, K_2^D = n_2) = {} & (1-d)^{n_2} p_1^{n_1} p_2^{n_2} \\
& - (1-d)^{n_2} p_1^{n_1+1} p_2^{n_2} \\
& - (1-d)^{n_2+1} p_1^{n_1} p_2^{n_2+1} \\
& + (1-d)^{n_2+1} p_1^{n_1+1} p_2^{n_2+1} \\
\\
= {} & (1-d)^{n_2} p_1^{n_1} p_2^{n_2} [1 - p_1 - (1-d)p_2 + (1-d)p_1 p_2] \\
= {} & (1-d)^{n_2} p_1^{n_1} p_2^{n_2} (1-p_1)[1 - (1-d)p_2].
\end{aligned} \tag{2}
$$

For $n_1 = n_2 (= 0, 1, ...)$, we have

$$
\begin{aligned}
P(K_1^D = n_1, K_2^D = n_1) = {} & (1-d)^{n_1} p_1^{n_1} p_2^{n_1} \\
& - (1-d)^{n_1+1} p_1^{n_1+1} p_2^{n_1} \\
& - (1-d)^{n_1+1} p_1^{n_1} p_2^{n_1+1} \\
& + (1-d)^{n_1+1} p_1^{n_1+1} p_2^{n_1+1} \\
\\
= {} & (1-d)^{n_1} p_1^{n_1} p_2^{n_1} [1 - (1-d)p_1 - (1-d)p_2 + (1-d)p_1 p_2)].
\end{aligned} \tag{3}
$$

For $n_1 = n_2 + 1, n_2 + 2, ...(n_2 = 0, 1, ...)$, we have

$$\begin{aligned} P(K_1^D = n_1, K_2^D = n_2) = (1-d)^{n_1} p_1^{n_1} p_2^{n_2} \\ - (1-d)^{n_1+1} p_1^{n_1+1} p_2^{n_2} \\ - (1-d)^{n_1} p_1^{n_1} p_2^{n_2+1} \\ + (1-d)^{n_1+1} p_1^{n_1+1} p_2^{n_2+1} \end{aligned} \quad (4)$$

$$\begin{aligned} &= (1-d)^{n_1} p_1^{n_1} p_2^{n_2} [1 - (1-d)p_1 - p_2 + (1-d)p_1 p_2] \\ &= (1-d)^{n_1} p_1^{n_1} p_2^{n_2} (1-p_2)[1-(1-d)p_1]. \end{aligned}$$

For additional verification see Appendix A.

Actuarial Present Value of Common Catastrophe

The joint life actuarial present value of one dollar payable at the end of the year of the first of either life x's or life y's death is denoted by A_{xy}. One would want this form of insurance to protect against the death of the wage earner. It is important to note that life y does not necessarily need to live until the end of the year. Also, life x corresponds to probabilities based off of individual 1 and life y corresponds to probabilities based off of individual 2.

In general the insurance is represented as

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} {}_k p_{xy} q_{x+k:y+k}.$$

${}_k p_{xy}$ is determined by the survival function $S_{K^D(x), K^D(y)}(n_1, n_2)$.

$$\begin{aligned} S_{K^D(x), K^D(y)}(n_1, n_2) &= P(K_1^D > n_1, K_2^D > n_2) \\ &= P(K(x) > n_1 \text{ and } K(y) > n_2 \text{ and } D > max(n_1, n_2)) \\ &= (1-d)^{max(n_1,n_2)+1} p_1^{n_1+1} p_2^{n_2+1}. \end{aligned}$$

Therefore using the above equations for the survival function we can determine ${}_k p_{xy}$

$$\begin{aligned} {}_k p_{xy} &= \frac{s(n_1 + k, n_2 + k)}{s(n_1, n_2)} \\ &= \frac{(1-d)^{max(n_1+k, n_2+k)+1} p_1^{n_1+k+1} p_2^{n_2+k+1}}{(1-d)^{max(n_1,n_2)+1} p_1^{n_1+1} p_2^{n_2+1}} \\ &= \frac{(1-d)^{max(n_1+k, n_2+k)+1} p_1^k p_2^k}{(1-d)^{max(n_1,n_2)+1}} \\ &= (1-d)^k p_1^k p_2^k. \end{aligned}$$

Hence,

$$q_{x+k:y+k} = 1 - p_{x+k:y+k} = 1 - (1-d)p_1 p_2.$$

Scaling by the starting ages to determine ${}_t p_{xy}$ works because the random variables K_1^D, K_2^D are still exponential in nature. Alternatively, we could calculate

$$\begin{aligned} {}_k p_{xy} q_{x+k:y+k} &= P(K_1^D > k-1, K_2^D > k-1) - P(K_1^D > k, K_2^D > k) \\ &= (1-d)^k p_1^k p_2^k - (1-d)^{k+1} p_1^{k+1} p_2^{k+1}. \end{aligned}$$

This conclusion we have already seen in our previous calculation.

We can also use the probabilities developed in the previous section to check our answer.

$$q_{x+k:y+k} = P(K_1^D = 0, K_2^D > 0) + P(K_1^D > 0, K_2^D = 0) + P(K_1^D = 0, K_2^D = 0).$$

Now using Equation (1)

$$\begin{aligned} P(K_1^D = 0, K_2^D > 0) &= P(K_1^D \geq 0, K_2^D > 0) - P(K_1^D > 0, K_2^D > 0) \\ &= P(K_2^D > 0) - P(K_1^D > 0, K_2^D > 0) \\ &= (1-d)p_1 - (1-d)p_1p_2. \end{aligned}$$

Similarly,

$$P(K_1^D > 0, K_2^D = 0) = (1-d)p_2 - (1-d)p_1p_2.$$

Therefore, from Equation (4)

$$P(K_1^D = 0, K_2^D = 0) = 1 - (1-d)p_1 - (1-d)p_2 + (1-d)p_1p_2.$$

Insert the required formulae into $q_{x+k:y+k}$

$$\begin{aligned} q_{x+k:y+k} &= (1-d)p_1 - (1-d)p_1p_2 + (1-d)p_2 - (1-d)p_1p_2 + 1 - (1-d)p_1 - (1-d)p_2 + (1-d)p_1p_2 \\ &= 1 - (1-d)p_1p_2 \\ &= 1 - p_{x+k:y+k}. \end{aligned}$$

We can now determine A_{xy}.

$$\begin{aligned} A_{xy} &= \sum_{k=0}^{\infty} v^{k+1}(1-d)^k p_1^k p_2^k [1 - (1-d)p_1p_2] \\ &= \sum_{k=0}^{\infty} v^{k+1}(1-d)^k p_1^k p_2^k - v^{k+1}(1-d)^{k+1} p_1^{k+1} p_2^{k+1} \\ &= \frac{v}{1 - v(1-d)p_1p_2} - \frac{v(1-d)p_1p_2}{1 - v(1-d)p_1p_2} \\ &= v\frac{1 - (1-d)p_1p_2}{1 - v(1-d)p_1p_2}. \end{aligned}$$

It is important to note that the smaller d is the greater the time until an individual's life is affected by the common disaster. When $d = 0$, then

$$A_{xy} = v\frac{1 - p_1p_2}{1 - vp_1p_2}.$$

When $d = 1$, then

$$A_{xy} = v\frac{1}{1} = v.$$

We can also calculate annuity values using $1 = r\ddot{a}_{xy} + A_{xy}$. Where $r = \frac{i}{1+i}$ usually denoted by discount d. Therefore, $\ddot{a}_{xy} = \frac{1 - A_{xy}}{r}$.

The last survivor status is payable at the end of the year upon the last death of either life x or life y. The actuarial present value of the last survivor status is denoted by $A_{\overline{xy}}$.

$$A_{\overline{xy}} = A_x + A_y - A_{xy}.$$

To determine this status we first need to develop the actuarial present values for lives x and y individually with the effect of the common disaster. The actuarial present value of one dollar payable at the end of the year is

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k}.$$

Therefore, we need

$$\begin{aligned} {}_kp_x &= \frac{s(n_1+k)}{s(n_1)} \\ &= \frac{(1-d)^{n_1+k} p_1^{n_1+k}}{(1-d)^{n_1} p_1^{n_1}} \\ &= (1-d)^k p_1^k. \end{aligned}$$

and

$${}_kq_x = 1-(1-d)^k p_1^k.$$

Now,

$$\begin{aligned} A_x &= \sum_{k=o}^{\infty} v^{k+1}(1-d)^k p_1^k [1-(1-d)p_2] \\ &= \sum_{k=0}^{\infty} v^{k+1}(1-d)^k p_1^k - v^{k+1}(1-d)^{k+1} p_1^{k+1} \\ &= \frac{v}{1-v(1-d)p_1} - \frac{v(1-d)p_1}{1-v(1-d)p_1} \\ &= v\frac{1-(1-d)p_1}{1-v(1-d)p_1}. \end{aligned}$$

Likewise,

$$A_y = v\frac{1-(1-d)p_2}{1-v(1-d)p_2}.$$

When $d=0$, we arrive at A_x with no disaster effect.

$$A_x = v\left[\frac{1-p_1}{1-vp_1}\right] = v\left[\frac{q_1}{1-vp_1}\right].$$

Therefore, inserting the required formulae into A_{xy}, we arrive at

$$A_{\overline{xy}} = v\left[\frac{1-(1-d)p_1}{1-v(1-d)p_1} + \frac{1-(1-d)p_2}{1-v(1-d)p_2} - \frac{1-(1-d)p_1p_2}{1-v(1-d)p_1p_2}\right].$$

When $d=0$, then we arrive at A_{xy} with no disaster effect.

$$A_{\overline{xy}} = v\left[\frac{1-p_1}{1-vp_1} + \frac{1-p_2}{1-vp_2} - \frac{1-p_1p_2}{1-vp_1p_2}\right].$$

When $d=1$, then the disaster definitely occurs and the payment is made at the end of the first period.

$$A_{\overline{xy}} = v\left[\frac{1}{1}+\frac{1}{1}-\frac{1}{1}\right] = v.$$

Life Lengthening

In this section we determine the probability of two individuals terminating due to biological causes specific to each life while a life lengthening force lengthens both lives. Therefore, we can separate the two forces acting on the lives into a biological aspect which seeks to terminate life and acts independent of the lives and a lengthening force which seeks to lengthen life. The life lengthening force is a way to separately account for decreased probability of biological death. For example, when looking at a group of smokers and non-smokers, one could consider those who do not smoke to have a lengthening force acting upon them, the fact that they do not smoke.

Define K_i as in the previous sections where the individual lives are independent and geometrically distributed. Let's assume that L is a lengthening force which can affect an individual by lengthening his or her life. Suppose that $P(L=k) = l^k(1-l)$ where $k = 0, 1, 2..., 0 < l < 1$. We assume that the lengthening force is a geometric random variable. l is the chance that the life continues. Therefore, the larger l is, the larger the probability of a life continuing due to lengthening. $(1-l)$ is the chance that the lengthening will fail in any one year. Define $K_i^L = K_i + L$ where $i = 1, 2, ...$ K_i^L are the whole number of years lived by individual i with lengthening. The lives K_i^L are dependent due to the lengthening force acting on both.

The PGF is the probability generating function. Knowing the probability generating function is equivalent to knowing the probability mass function which we wish to develop. The PGF can be defined as

$$\begin{aligned} PGF(x,y) &= E\left[x^{K_1^L} y^{K_2^L}\right] \\ &= E\left[x^{K_1+L} y^{K_2+L}\right] \\ &= E\left[x^{K_1}\right] E\left[y^{K_2}\right] E\left[(xy)^L\right] \\ &= \sum_{k=0}^{\infty} x^k p_1^k q_1 \sum_{k=0}^{\infty} y^k p_2^k q_2 \sum_{k=0}^{\infty} (xy)^k (1-l) l^k \\ &= \frac{q_1}{1-p_1 x} \frac{q_2}{1-p_2 y} \frac{(1-l)}{1-lxy}. \end{aligned}$$

Now sum over the partial derivatives at $x, y = 0$ to solve for the probability mass function

$$\frac{\partial^{n_1+n_2} P(0,0)}{\partial y^{n_2} \partial x^{n_1}} = n_1! n_2! P(K_1^L = n_1, K_2^L = n_2)$$

$$= \frac{\partial^{n_2}}{\partial y^{n_2}} \left[\frac{\partial^{n_1} P(x,y)}{\partial x^{n_1}}\right] |_{x=y=0}.$$

Split into separate cases

$$= \frac{\partial^{n_2}}{\partial y^{n_2}} \left[\left(\sum_{k=0}^{\infty} y^k p_2^k q_2\right) \sum_{m=0}^{n_1} \frac{n_1!}{(n_1-m)!m!} \right.$$
$$\left. \frac{\partial^m}{\partial x^m} \left(\sum_{k=0}^{\infty} x^k p_1^k q_1\right) \frac{\partial^{n_1-m}}{\partial x^{n_1-m}} \left(\sum_{k=0}^{\infty} (xy)^k l^k (1-l)\right)\right] |_{x=y=0}$$

Solve the last summations

$$= \frac{\partial^{n_2}}{\partial y^{n_2}} \left[\left(\sum_{k=0}^{\infty} y^k p_2^k q_2\right) \sum_{m=0}^{n_1} \frac{n_1!}{(n_1-m)!m!} m! p_1^m q_1 \right.$$
$$\left. \sum_{k=n_1-m}^{\infty} k(k-1)...(k-n_1+m+1) x^{k-n_1+m} y^k l^k (1-l)\right] |_{x=y=0}$$

$$= \frac{\partial^{n_2}}{\partial y^{n_2}} \left[\left(\sum_{k=0}^{\infty} y^k p_2^k q_2 \right) \sum_{m=0}^{n_1} \frac{n_1!}{(n_1-m)!m!} m! p_1^m q_1 (n_1-m)! y^{n_1-m} l^{n_1-m} (1-l) \right] |_{y=o}$$

$$= \frac{\partial^{n_2}}{\partial y^{n_2}} \left[\left(\sum_{k=0}^{\infty} y^k p_2^k q_2 \right) \sum_{m=0}^{n_1} n_1! p_1^m q_1 y^{n_1-m} l^{n_1-m} (1-l) \right] |_{y=0}$$

Split into cases again

$$= \sum_{r=0}^{n_2} \frac{n_2!}{(n_2-r)!r!} \frac{\partial^r}{\partial y^r} \left(\sum_{k=0}^{\infty} y^k p_2^k q_2 \right) \frac{\partial^{n_2-r}}{\partial y^{n_2-r}} \left(\sum_{m=0}^{n_1} n_1! p_1^m q_1 l^{n_1-m} (1-l) y^{n_1-m} \right) |_{y=0}$$

$$= \sum_{r=0}^{n_2} \frac{n_2!}{(n_2-r)!r!} r! p_2^r q_2 \frac{\partial^{n_2-r}}{\partial y^{n_2-r}} \sum_{m=0}^{n_1} n_1! p_1^{n_1-m} q_1 l^m (1-l) y^m |_{y=0}$$

Simplify,

$$= \sum_{r=0}^{n_2} \frac{n_2!}{(n_2-r)!r!} (n_2-r)! p_2^{n_2-r} q_2 \frac{\partial^r}{\partial y^r} \sum_{m=0}^{n_1} n_1! p_1^{n_1-m} q_1 l^m (1-l) y^m |_{y=0}$$

$$= \sum_{r=0}^{n_2} \frac{n_2!}{(n_2-r)!r!} (n_2-r)! p_2^{n_2-r} q_2 [n_1! p_1^{n_1-r} q_1 l^r (1-l) r! 1_{r,n_1}]$$

Reduce the summations

$$= \sum_{r=0}^{min(n_1,n_2)} n_1! n_2! p_2^{n_2-r} q_2 P_1^{n_1-r} q_1 l^r (1-l)$$

$$= n_1! n_2! q_1 q_2 (1-l) \sum_{r=0}^{min(n_2,n_1)} p_1^{n_1-r} p_2^{n_2-r} l^r$$

$$= n_1! n_2! q_1 q_2 (1-l) \frac{p_1^{n_1} p_2^{n_2} - P_1^{n_1-min(n_1,n_2)-1} p_2^{n_2-min(n_1,n_2)-1} l^{min(n_1,n_2)+1}}{1-\frac{l}{p_1 p_2}}$$

Therefore,

$$P(K_1^L = n_1, K_2^L = n_2) = q_1 q_2 (1-l) \frac{p_1^{n_1+1} p_2^{n_2+1} - p_1^{n_1-min(n_1,n_2)} p_2^{n_2-min(n_1,n_2)} l^{min(n_1,n_2)+1}}{p_1 p_2 - l}. \tag{5}$$

For $n_1 \le n_2 (n_1 = 0, 1, ...n_2)$, we have

$$\begin{aligned} P(K_1^L = n_1, K_2^L = n_2) &= q_1 q_2 (1-l) \frac{p_1^{n_1+1} p_2^{n_2+1} - p_1^{n_1-n_1} p_2^{n_2-n_1} l^{n_1+1}}{p_1 p_2 - l} \\ &= q_1 q_2 (1-l) \frac{p_1^{n_1+1} p_2^{n_2+1} - p_2^{n_2-n_1} l^{n_1+1}}{p_1 p_2 - l}. \end{aligned} \tag{6}$$

For $n_1 = n_2 + 1, n_2 + 2, ...$

$$P(K_1^L = n_1, K_2^L = n_2) = q_1 q_2 (1-l) \frac{p_1^{n_1+1} p_2^{n_2+1} - p_1^{n_1-n_2} l^{n_2+1}}{p_1 p_2 - l}. \tag{7}$$

For additional verification see Appendix B.

Actuarial Present Value of Life Lengthening

The joint life actuarial present value of one dollar payable at the end of the year of the first of either life x's or life y's death is denoted by A_{xy}. One would want this form of insurance to protect against the death of the wage earner. It is important to note that life y does not necessarily need to live until the end of the year.

In general the insurance is represented as

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1}{}_kp_{xy}q_{x+k:y+k}.$$

Now, if we try to develop ${}_kp_{xy}$ by scaling by the starting age as we did for the common disaster we arrive at

$${}_kp_{xy} = \frac{(1-l)(p_1p_2)^{n+k+2} - (1-p_1p_2)l^{n+k+2}}{(1-l)(p_1p_2)^{n+2} - (1-p_1p_2)l^{n+2}}.$$

With life lengthening, the starting age, for example the marriage date, of the lengthening force is very important. ${}_kp_{xy}$ is not memoryless.

For simplicity we will determine ${}_kp_{xy}q_{x+k:y+k} = P(K_1^L > k-1, K_2^L > k-1) - P(K_1^L > k, K_2^L > k)$. Therefore, we need to determine $P(K_1^L > n, K_2^L > n)$. Use Equation (5) in the previous section. For $n_1 > n_2$

$$\begin{aligned}
P(K_1^L > n_1, K_2^L > n_2) &= \sum_{k_2=n_2+1}^{\infty}\sum_{k_1=n_1+1}^{\infty} P(K_1^L = k_1, K_2^L = k_2) \\
&= \sum_{n_1+1\le k_1\le k_2}^{\infty} P(K_1^L = k_1, K_2^L = k_2) + \sum_{n_2+1\le k_2\le k_1}^{\infty} P(K_1^L = k_1, K_2^L = k_2) \\
&= \sum_{k_2=n_1+1}^{\infty}\sum_{k_1=n_1+1}^{k_2} q_1q_2(1-l)\frac{p_1^{k_1+1}p_2^{k_2+1} - p_2^{k_2-k_1}l^{k_1+1}}{p_1p_2-l} \\
&\quad + \sum_{k_1=n_1+1}^{\infty}\sum_{k_2=n_2+1}^{k_1-1} q_1q_2(1-l)\frac{p_1^{k_1+1}p_2^{k_2+1} - p_1^{k_1-k_2}l^{k_2+1}}{p_1p_2-l}
\end{aligned}$$

Split into cases

$$\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\sum_{k_2=n_1+1}^{\infty} p_2^{k_2+1}\sum_{k_1=n_1+1}^{k_2} p_1^{k_1+1} - \sum_{k_2=n_1+1}^{\infty} p_2^{k_2}l\sum_{k_1=n_1+1}^{k_2}\left(\frac{l}{p_2}\right)^{k_1}\right. \\
&\qquad \left. + \sum_{k_1=n_1+1}^{\infty} p_1^{k_1+1}\sum_{k_2=n_2+1}^{k_1-1} p_2^{k_2+1} - \sum_{k_1=n_1+1}^{\infty} p_1^{k_1}l\sum_{k_2=n_2+1}^{k_1-1}\left(\frac{l}{p_1}\right)^{k_2}\right]
\end{aligned}$$

Solve the inner summations

$$\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\sum_{k_2=n_1+1}^{\infty} p_2^{k_2+1}\frac{p_1^{n_1+2}-p_1^{k_2+2}}{1-p_1} - \sum_{k_2=n_1+1}^{\infty} p_2^{k_2}l\frac{\left(\frac{l}{p_1}\right)^{n_1+1} - \left(\frac{l}{p_1}\right)^{k_2+2}}{1-\frac{l}{p_2}}\right. \\
&\qquad \left. + \sum_{k_1=n_1+1}^{\infty} p_1^{k_1+1}\frac{p_2^{n_2+2}-p_2^{k_1+1}}{1-p_2} - \sum_{k_1=n_1+1}^{\infty} p_1^{k}l\frac{\left(\frac{l}{p_1}\right)^{n_2+1} - \left(\frac{l}{p_1}\right)^{k_1}}{1-\frac{l}{p_1}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{1}{q_1}p_1^{n_1+2}p_2\sum_{k_2=n_1+1}^{\infty}p_2^{k_2} - \frac{1}{q_1}p_2p_1^2\sum_{k_2=n_1+1}^{\infty}(p_1p_2)^{k_2} \\
&\quad - \frac{p_2l}{p_2-l}\left(\frac{l}{p_2}\right)^{n_1+1}\sum_{k_2=n_1+1}^{\infty}p_2^{k_2} + \frac{p_2l}{p_2-l}\frac{l}{p_2}\sum_{k_2=n_1+1}^{\infty}l^{k_2} \\
&\quad + \frac{1}{q_2}p_2^{n_2+2}p_1\sum_{k_1=n_1+1}^{\infty}p_1^{k_1} - \frac{1}{q_2}p_1p_2\sum_{k_1=n_1+1}^{\infty}(p_1p_2)^{k_1} \\
&\quad - \frac{p_1l}{p_1-l}\left(\frac{l}{p_1}\right)^{n_2+1}\sum_{k_1=n_1+1}^{\infty}p_1^{k_1} + \frac{p_1l}{p_1-l}\sum_{k_1=n_1+1}^{\infty}l^{k_1}\Bigg]
\end{aligned}
\tag{8}
$$

Solve the last summations

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{1}{q_1}p_1^{n_1+2}p_2\frac{p_2^{n_1+1}}{1-p_2} - \frac{1}{q_1}p_2p_1^2\frac{(p_1p_2)^{n_1+1}}{1-p_1p_2} - \frac{p_2l}{p_2-l}\left(\frac{l}{p_2}\right)^{n_1+1}\frac{p_2^{n_1+1}}{1-p_2} + \frac{l^2}{p_2-1}\frac{l^{n_1+1}}{1-l} \\
&\quad + \frac{p_1p_2^{n_2+2}}{q_2}\frac{p_1^{n_1+1}}{1-p_1} - \frac{p_1p_2}{q_2}\frac{(p_1p_2)^{n_1+1}}{1-p_1p_2} - \frac{p_1l}{p_1-l}\left(\frac{l}{p_1}\right)^{n_2+1}\frac{p_1^{n_1+1}}{1-p_1} + \frac{p_1l}{p_1-l}\frac{l^{n_1+1}}{1-l}\Bigg]
\end{aligned}
$$

Simplify,

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{p_1^{n_1+2}p_2^{n_1+2}}{q_1q_2} - \frac{p_1^{n_1+3}p_2^{n_1+2}}{q_1(1-p_1p_2)} - \frac{p_2l^{n_1+2}}{q_2(p_2-l)} + \frac{l^{n_1+3}}{(1-l)(p_2-l)} \\
&\quad + \frac{p_1^{n_1+2}p_2^{n_2+2}}{q_1q_2} - \frac{p_1^{n_1+2}p_2^{n_1+2}}{q_2(1-p_1p_2)} - \frac{p_1l^{n_2+2}}{q_1(p_1-l)} + \frac{p_1l^{n_1+2}}{(1-l)(p_1-l)}\Bigg]
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{p_1^{n_1+2}p_2^{n_2+2}(1+p_2^{n_1-n_2})}{q_1q_2} - \frac{(1-p_2)p_1^{n_1+3}p_2^{n_1+2}+(1-p_1)p_1^{n_1+2}p_2^{n_1+2}}{(1-l)(p_1-l)(p_2-l)} \\
&\quad - \frac{(1-p_1)p_2(p_1-l)l^{n_1+2}+(1-p_2)p_1(p_2-l)l^{n_2+2}}{q_1q_2(p_1-l)(p_2-l)} + \frac{(p_1-l)l^{n_1+3}+(p_2-l)p_1l^{n_1+2}}{(1-l)(p_1-l)(p_2-l)}\Bigg]
\end{aligned}
$$

Group like terms

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{p_1^{n_1+2}p_2^{n_2+2}(1+p_2^{n_1-n_2})}{q_1q_2} - \frac{p_1^{n_1+2}p_2^{n_1+2}(1-p_1p_2)}{q_1q_2(1-p_1p_2)} \\
&\quad - \frac{p_2l^{n_1+2}}{q_2(p_2-l)} - \frac{p_1l^{n_2+2}}{q_1(p_1-l)} + \frac{l^{n_1+2}(p_1p_2-l^2)}{(1-l)(p_1-l)(p_2-l)}\Bigg]
\end{aligned}
$$

$$
= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\frac{p_1^{n_1+2}p_2^{n_2+2}}{q_1q_2} - \frac{p_2l^{n_1+2}}{q_2(p_2-l)} - \frac{p_1l^{n_2+2}}{q_1(p_1-l)} + \frac{l^{n_1+2}(p_1p_2-l^2)}{(1-l)(p_1-l)(p_2-l)}\right].
$$

For $n_2 > n_1$

$$
= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\frac{p_1^{n_1+2}p_2^{n_2+2}}{q_1q_2} - \frac{p_1l^{n_2+2}}{q_1(p_1-l)} - \frac{p_2l^{n_1+2}}{q_2(p_2-l)} + \frac{l^{n_2+2}(p_1p_2-l^2)}{(1-l)(p_1-l)(p_2-l)}\right].
$$

For $n_1 = n_2 = n$

$$
\begin{aligned}
P(K_1^L > n, K_2^L > n) &= \sum_{n+1\le k_1\le k_2}^{\infty}P(K_1^L = k_1, K_2^L = k_2) + \sum_{n+1\le k_2\le k_1}^{\infty}P(K_1^L = k_1, K_2^L = k_2) \\
&= \sum_{k_2=n+1}^{\infty}\sum_{k_1=n+1}^{k_2}q_1q_2(1-l)\frac{p_1^{k_1+1}p_2^{k_2+1}-p_2^{k_2-k_1}l^{k_1+1}}{p_1p_2-l} \\
&\quad + \sum_{k_1=n+2}^{\infty}\sum_{k_2=n+1}^{k_1-1}q_1q_2(1-l)\frac{p_1^{k_1+1}p_2^{k_2+1}-p_1^{k_1-k_2}l^{k_2+1}}{p_1p_2-l}
\end{aligned}
$$

From Equation (8) and Equation (9)

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{p_1^{n+2}p_2^{n+2}}{q_1q_2} - \frac{p_1^{n+3}p_2^{n+2}}{q_1(1-p_1p_2)} - \frac{p_2 l^{n+2}}{q_2(p_2-l)} + \frac{l^{n+3}}{(1-l)(p_2-l)}$$
$$+\frac{1}{q_2}p_2^{n+2}p_1\sum_{k_1=n+2}^{\infty}p_1^{k_1} - \frac{1}{q_2}p_1p_2\sum_{k_1=n_1+2}^{\infty}(p_1p_2)^{k_1}$$
$$-\frac{p_1l}{p_1-l}\left(\frac{l}{p_1}\right)^{n+1}\sum_{k_1=n+2}^{\infty}p_1^{k_1} + \frac{p_1l}{p_1-l}\sum_{k_1=n+2}^{\infty}l^{k_1}\Bigg]$$

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{p_1^{n+2}p_2^{n+2}}{q_1q_2} - \frac{p_1^{n+3}p_2^{n+2}}{q_1(1-p_1p_2)} - \frac{p_2l^{n+2}}{q_2(p_2-l)} + \frac{l^{n+3}}{(1-l)(p_2-l)}$$
$$+\frac{p_1^{n+3}p_2^{n+2}}{q_1q_2} - \frac{p_1^{n+3}p_2^{n+3}}{q_2(1-p_1p_2)} - \frac{p_1^2l^{n+2}}{q_1(p_1-l)} + \frac{p_1l^{n+3}}{(1-l)(p_1-l)}\Bigg]$$

Group over common denominators

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{(p_1p_2)^{n+2}(1+p_1)}{q_1q_2} - \frac{(1-p_2)p_1^{n+3}p_2^{n+2} + (1-p_1)p_1^{n+3}p_2^{n+3}}{q_1q_2(1-p_1p_2)}$$
$$-\frac{p_2(1-p_1)(p_1-l) + p_1^2(1-p_2)(p_2-l)}{q_1q_2(p_1-l)(p_2-l)}l^{n+2} + \frac{(p_1-l)+p_1(p_2-l)}{(1-l)(p_1-l)(p_2-l)}l^{n+3}\Bigg]$$

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{(p_1p_2)^{n+2}(1+p_1)}{q_1q_2} - \frac{p_1^{n+3}p_2^{n+2} - p_1^{n+4}p_2^{n+3}}{q_1q_2(1-p_1p_2)}$$
$$-\frac{p_2(p_1-p_1^2-l+p_1l) + p_1^2(p_2-p_2^2-l+p_2l)}{q_1q_2(p_1-l)(p_2-l)}l^{n+2} + \frac{p_1-l+p_1p_2-p_1l}{(1-l)(p_1-l)(p_2-l)}l^{n+3}\Bigg]$$

Simplify numerators

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{(p_1p_2)^{n+2}(1+p_1)}{q_1q_2} - \frac{(p_1p_2)^{n+2}p_1(1-p_1p_2)}{q_1q_2(1-p_1p_2)} - \frac{p_1p_2(1-p_1p_2)l^{n+2}}{q_1q_2(p_1-l)(p_2-l)}$$
$$-\frac{-p_2+p_1p_2-p_1^2+p_2p_1^2}{q_1q_2(p_1-l)(p_2-l)}l^{n+3} + \frac{p_1+p_1p_2}{(1-l)(p_1-l)(p_2-l)}l^{n+3} - \frac{1+p_1)l}{(1-l)(p_1-l)(p_2-l)}l^{n+3}\Bigg]$$

$$= \frac{q_1q_2(1-l)}{p_1p_2-l}\Bigg[\frac{(p_1p_2)^{n+2}}{q_1q_2} - \frac{p_1p_2(1-p_1p_2)l^{n+2}}{q_1q_2(p_1-l)(p_2-l)(1-l)}$$
$$+\frac{p_1p_2(1-p_1p_2)l^{n+3}}{q_1q_2(p_1-l)(p_2-l)(1-l)} + \frac{p_2-p_1p_2+p_1^2-p_1^2p_2}{q_1q_2(p_1-l)(p_2-l)(1-l)}l^{n+3}$$
$$-\frac{p_2-p_1p_2+p_1^2-p_1^2p_2}{q_1q_2(p_1-l)(p_2-l)(1-l)}l^{n+4} + \frac{p_1(1+p_2)(1-p_1)(1-p_2)}{q_1q_2(p_1-l)(p_2-l)(1-l)}l^{n+3}$$
$$-\frac{(1+p_1)(1-p_1)(1-p_2)}{q_1q_2(p_1-l)(p_2-l)(1-l)}l^{n+4}\Bigg]$$

Cancel terms

$$= \frac{1-l}{p_1p_2-l}\Bigg[(p_1p_2)^{n+2} - \frac{p_1p_2(1-p_1p_2)l^{n+2}}{(p_1-l)(p_2-l)(1-l)}$$
$$+\frac{p_1p_2(1-p_1p_2)+p_2-p_1p_2+p_1^2-p_1^2p_2+p_1(1-p_1)(1-p_2^2)}{(1-l)(p_1-l)(p_2-l)}l^{n+3}$$
$$-\frac{p_2-p_1p_2+p_1^2-p_1^2p_2+(1-p_2)(1-p_1^2)}{(1-l)(p_1-l)(p_2-l)}l^{n+4}\Bigg]$$

$$= \frac{1-l}{p_1p_2-l}\left[(p_1p_2)^{n+2} - \frac{p_1p_2(1-p_1p_2)l^{n+2}}{(p_1-l)(p_2-l)(1-l)} + \frac{p_1+p_2-p_1p_2(p_1+p_2)}{(p_1-l)(p_2-l)(1-l)}l^{n+3} - \frac{1-p_1p_2}{(p_1-l)(p_2-l)(1-l)}l^{n+4}\right]$$

Simplify,

$$= \frac{1-l}{p_1p_2-l}\left[(p_1p_2)^{n+2} - \frac{(1-p_1p_2)l^{n+2}}{(1-l)(p_1-l)(p_2-l)}\left(p_1p_2-(p_1+p_2)l+l^2\right)\right]$$
$$= \frac{1-l}{p_1p_2-l}\left[(p_1p_2)^{n+2} - \frac{(1-p_1p_2)l^{n+2}}{(1-l)}\right]$$
$$= \frac{1}{p_1p_2-l}\left[(1-l)(p_1p_2)^{n+2} - (1-p_1p_2)l^{n+2}\right].$$

For $n=0$, we have

$$= \frac{1}{p_1p_2-l}\left[(1-l)(p_1p_2)^2 - (1-p_1p_2)l^2\right]$$
$$= \frac{1}{p_1p_2-l}\left[(p_1p_2)^2 - (p_1p_2)^2l - l^2 + (p_1p_2)l^2\right]$$
$$= \frac{1}{p_1p_2-l}\left[p_1p_2l(l-p_1p_2) + (p_1p_2-l)(p_1p_2+l)\right]$$
$$= p_1p_2 + l - p_1p_2l$$
$$= 1-(1-l)(1-p_1p_2).$$

Again this is the same as $P(K_1^L > 0, K_2^L > 0)$.

$$\begin{aligned} P(K_1^L > 0, K_2^L > 0) &= 1 - P(K_1^L = 0 \text{ or } K_2^L = 0) \\ &= 1 - \left[P(K_1^L = 0, K_2^L > 0) + P(K_1^L > 0, K_2^L = 0) + P(K_1^L = 0, K_2^L = 0)\right] \\ &= 1 - \left[(1-l)(1-p_1)p_2 + (1-l)p_1(1-p_2) + (1-l)(1-p_1)(1-p_2)\right] \\ &= 1 - (1-l)\left[p_2 - p_1p_2 + p_1 - p_1p_2 + 1 - p_1 - p_2 + p_1p_2\right] \\ &= 1-(1-l)(1-p_1p_2). \end{aligned}$$

Now, inserting the required formulae into ${}_kp_{xy}q_{x+k:y+k}$

$$\begin{aligned} {}_kp_{xy}q_{x+k:y+k} &= P(K_1^L > k-1, K_2^L > k-1) - P(K_1^L > k, K_2^L > k) \\ &= \frac{1}{p_1p_2-l}\left[(1-l)(p_1p_2)^{k+1} - (1-p_1p_2)l^{k+1} - (1-l)(p_1p_2)^{k+2} + (1-p_1p_2)l^{k+2}\right] \\ &= \frac{1}{p_1p_2-l}\left[(1-l)\left((p_1p_2)^{k+1} - (p_1p_2)^{k+2}\right) - (1-p_1p_2)\left(l^{k+1}-l^{k+2}\right)\right] \\ &= \frac{1}{p_1p_2-l}\left[(1-l)(p_1p_2)^{k+1}(1-p_1p_2) - (1-p_1p_2)l^{k+1}(1-l)\right] \\ &= \frac{(1-l)(1-p_1p_2)}{p_1p_2-l}\left[(p_1p_2)^{k+1} - l^{k+1}\right]. \end{aligned}$$

Inserting the required formulae into A_{xy}, we arrive at

$$\begin{aligned} A_{xy} &= \frac{(1-l)(1-p_1p_2)}{p_1p_2-l}\sum_{k=0}^{\infty} v^{k+1}\left[(p_1p_2)^{k+1} - l^{k+1}\right] \\ &= \frac{(1-l)(1-p_1p_2)}{p_1p_2-l}\left[\frac{vp_1p_2}{1-vp_1p_2} - \frac{vl}{1-vl}\right] \\ &= \frac{v(1-l)(1-p_1p_2)}{p_1p_2-l}\left[\frac{p_1p_2}{1-vp_1p_2} - \frac{l}{1-vl}\right]. \end{aligned}$$

It is important to note that the smaller l is the less the lives are affected by the lengthening. When $l = 0$, or when there is no lengthening, we arrive at the same actuarial present value as when $d = 0$.

$$\begin{aligned} A_{xy} &= v\frac{1 - p_1p_2}{p_1p_2} * \frac{p_1p_2}{1 - vp_1p_2} \\ &= v\frac{1 - p_1p_2}{1 - vp_1p_2}. \end{aligned}$$

When $l = 1$, then we arrive at $A_{xy} = 0$. The lengthening will cause you to live forever.

We can also calculate annuity values using $1 = r\ddot{a}_{xy} + A_{xy}$. Where $r = \frac{i}{1+i}$ usually denoted by discount d. Therefore, $\ddot{a}_{xy} = \frac{1 - A_{xy}}{r}$.

The last survivor status is payable at the end of the year upon the last death of either life x or life y. The actuarial present value of the last survivor status is denoted by $A_{\overline{xy}}$.

$$A_{\overline{xy}} = A_x + A_y - A_{xy}.$$

To determine this status we first need to develop the actuarial present values for lives x and y individually with the effect of the life lengthening. The actuarial present value of one dollar payable at the end of the year is

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k}.$$

In determining ${}_kp_x q_{x+k} = P(K_1^L > k - 1) - P(K_1^L > k)$ we need $P(K_1^L > n)$.

$$P(L = k) = l^k(1 - l).$$

$$P(K_1 = k) = p_1^k q_1.$$

Hence,

$$P(L + K_1 = k) = \sum_{m=0}^{k} l^m(1 - l)p_1^{k-m}q_1 = (1 - l)p_1^k q_1 \frac{1 - l^{k+1}\frac{q_1}{p_1}}{1 - \frac{l}{p_1}}.$$

and

$$\begin{aligned} P(K_1^L > n) &= \sum_{r=n+1}^{\infty} (1 - l)p_1^r q_1 \frac{1 - l^{r+1}\frac{q_1}{p_1}}{1 - \frac{l}{p_1}} \\ &= \frac{(1 - l)q_1}{1 - \frac{l}{p_1}} \left[\sum_{r=n+1}^{\infty} p_1^r - q_1 \sum_{r=n+1}^{\infty} l^{r+1}p_1^{r-1} \right] \\ &= \frac{(1 - l)q_1}{1 - \frac{l}{p_1}} \left[\frac{p_1^{n+1}}{1 - p_1} - q_1 \frac{l^{n+2}p_1^n}{1 - lp_1} \right]. \end{aligned} \tag{10}$$

Therefore, using Equation (10)

$$\begin{aligned} {}_kp_x q_{x+k} &= \frac{(1 - l)q_1}{1 - \frac{l}{p_1}} \left[\frac{p_1^k}{1 - p_1} - q_1 \frac{l^{k+1}p_1^{k-1}}{1 - lp_1} - \frac{p_1^{k+1}}{1 - p_1} + q_1 \frac{l^{k+2}p_1^k}{1 - lp_1} \right] \\ &= \frac{(1 - l)q_1}{1 - \frac{l}{p_1}} \left[p_1^k - q_1 l^{k+1} p_1^{k-1} \right]. \end{aligned}$$

Inserting the required formulae into A_x, we arrive at

$$\begin{aligned}A_x &= \frac{(1-l)q_1}{1-\frac{l}{p_1}}\sum_{k=o}^{\infty} v^{k+1}\left[p_1^k - q_1 l^{k+1} p_1^{k-1}\right] \\ &= \frac{(1-l)q_1}{1-\frac{l}{p_1}}\left[\frac{v}{1-vp_1} - q_1\frac{v\frac{l}{p_1}}{1-vlp_1}\right] \\ &= v\frac{(1-l)q_1}{p_1-l}\left[\frac{p_1}{1-vp_1} - \frac{lq_1}{1-vlp_1}\right].\end{aligned}$$

Similarly,

$$A_y = v\frac{(1-l)q_2}{p_2-l}\left[\frac{p_2}{1-vp_2} - \frac{lq_2}{1-vlp_2}\right].$$

When $l = 0$, we reach the same conclusion as when $d = 0$.

$$A_x = vq_1\frac{1}{1-vp_1}.$$

Finally,

$$\begin{aligned}A_{\overline{xy}} = v\Bigg[&\frac{(1-l)q_1}{p_1-l}\left(\frac{p_1}{1-vp_1} - \frac{lq_1}{1-vlp_1}\right) \\ &+ \frac{(1-l)q_2}{p_2-l}\left(\frac{p_2}{1-vp_2} - \frac{lq_2}{1-vlp_2}\right) \\ &- \frac{(1-l)(1-p_1p_2)}{p_1p_2-l}\left(\frac{p_1p_2}{1-vp_1p_2} - \frac{l}{1-vl}\right)\Bigg].\end{aligned}$$

Since $A_{\overline{xy}}$ is linear when $l = 0$ and $d = 0$, we arrive at the same conclusions as before.

Common Disaster and Life Lengthening

In this section we determine the probability of two individuals terminating due to biological causes or a common disaster while having their lives lengthened by a common lengthening force. Therefore, we can look at three separate forces acting on each life, the biological, the common disaster, and the life lengthening. The individual lives are independent of each other,however, once the common disaster and life lengthening force are accounted for the individuals are dependent due to the disaster and lengthening which are acting on both. By using both the common disaster and the life lengthening one can account for those factors both increasing and decreasing biological mortality.

We will use the same definitions used to develop the separate cases involving a common disaster and life lengthening in the previous two sections. We will define $(K_i^L)^D$ as the number of years lived by an individual i with the life lengthening force while being subjected to a common disaster.

$$(K_1^L)^D = min(K_1^L, D) = min(K_1 + L, D).$$

Therefore,

$$\begin{aligned}P(K_1^{LD} > n_1, K_2^{LD} > n_2) &= P(D > n_1, D > n_2, K_1^L > n_1, K_2^L > n_2) \\ &= P(D > max(n_1, n_2)) * P(K_1^L > n_1, K_2^L > n_2) \\ &= (1-d)^{max(n_1,n_2)+1} * P(K_1^L > n_1, K_2^L > n_2).\end{aligned}$$

As in the previous sections we can set $n_1 = n_2 = n$ since their initial ages do not matter. Only the start of the life lengthening matters which we will assume starts at n.

Hence,

$$P(K_1^{LD} > n, K_2^{LD} > n) = (1-d)^{n+1} * \frac{1}{p_1p_2 - l}\left[(1-l)(p_1p_2)^{n+2} - (1-p_1p_2)l^{n+2}\right]. \quad (11)$$

Actuarial Present Value of Common Disaster and Life Lengthening

The joint life actuarial present value of one dollar payable at the end of the year of the first of either life x's or life y's death is denoted by A_{xy}. One would want this form of insurance to protect against the death of the wage earner. It is important to note that life y does not necessarily need to live until the end of the year.

In general the insurance is represented as

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} {}_kp_{xy} q_{x+k:y+k}.$$

Once again we need ${}_kp_{xy}q_{x+k:y+k}$ which we determine using Equation (11).

$$\begin{aligned}
{}_kp_{xy}q_{x+k:y+k} &= P(K_1^{LD} > k-1, K_2^{LD} > k-2) - P(K_1^{LD} > k, K_2^{LD} > k) \\
&= \frac{1}{p_1p_2 - l}\left[(1-l)(1-d)^k(p_1p_2)^{k+1} - (1-p_1p_2)(1-d)^k l^{k+1}\right. \\
&\quad \left. -(1-l)(1-d)^{k+1}(p_1p_2)^{k+2} + (1-p_1p_2)(1-d)^{k+1}l^{k+2}\right] \\
&= \frac{1}{p_1p_2 - l}\left[(1-l)(1-d)^k(p_1p_2)^{k+1}(1-(1-d)p_1p_2)\right. \\
&\quad \left. -(1-p_1p_2)(1-d)^k l^{k+1}(1-(1-d)l)\right].
\end{aligned}$$

Inserting the required formulae into A_{xy}, we arrive at

$$\begin{aligned}
A_{xy} &= \frac{1}{p_1p_2 - l}\left[(1-l)(1-(1-d)p_1p_2)\sum_{k=0}^{\infty} v^{k+1}(1-d)^k(p_1p_2)^{k+1}\right. \\
&\quad \left. -(1-p_1p_2)(1-(1-d)l)\sum_{k=0}^{\infty} v^{k+1}(1-d)^k l^{k+1}\right] \\
&= \frac{1}{p_1p_2 - l}\left[(1-l)(1-(1-d)p_1p_2)\frac{vp_1p_2}{1-v(1-d)p_1p_2} - (1-p_1p_2)(1-(1-d)l)\frac{vl}{1-v(1-d)l}\right].
\end{aligned}$$

When $d = 0$, you arrive at the actuarial present value with only the lengthening.

$$A_{xy} = \frac{1}{p_1p_2 - l}\left[(1-l)(1-p_1p_2)\frac{vp_1p_2}{1-vp_1p_2} - (1-p_1p_2)(1-l)\frac{vl}{1-vl}\right].$$

When $l = 0$, you arrive at the actuarial present value with only the disaster.

$$A_{xy} = \frac{1}{p_1p_2}\left[(1-(1-d)p_1p_2)\frac{vp_1p_2}{1-v(1-d)p_1p_2}\right] = v\left[\frac{1-(1-d)p_1p_2}{1-v(1-d)p_1p_2}\right].$$

We can also calculate annuity values using $1 = r\ddot{a}_{xy} + A_{xy}$. Where $r = \frac{i}{1+i}$ usually denoted by discount d. Therefore, $\ddot{a}_{xy} = \frac{1-A_{xy}}{r}$.

The last survivor status is payable at the end of the year upon the last death of either life x or life y. The actuarial present value of the last survivor status is denoted by $A_{\overline{xy}}$.

$$A_{\overline{xy}} = A_x + A_y - A_{xy}.$$

To determine this status we first need to develop the actuarial present values for lives x and y individually with the effect of the common disaster and life lengthening. The actuarial present value of one dollar payable at the end of the year is

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k}.$$

Therefore, we need

$$ {}_kp_xq_{x+k} = P(K_1^{LD} > k-1) - P(K_1^{LD} > k). $$

Now,

$$ P(K_1^{LD} > n) = P(D > n, K_1^L > n) = (1-d)^{n+1}\frac{(1-l)q_1}{1-\frac{l}{p_1}}\left[\frac{p_1^{n+1}}{1-p_1} - q_1\frac{l^{n+2}p_1^n}{1-lp_1}\right]. $$

Therefore,

$$ \begin{aligned} {}_kp_xq_{x+k} &= \frac{(1-l)q_1}{1-\frac{l}{p_1}}\left[\frac{(1-d)^kp_1^k}{1-p_1} - q_1\frac{(1-d)^kl^{k+1}p_1^{k-1}}{1-lp_1}\right. \\ &\quad \left. - \frac{(1-d)^{k+1}p_1^{k+1}}{1-p_1} - q_1\frac{(1-d)^{k+1}l^{k+2}p_1^k}{1-lp_1}\right] \\ &= \frac{(1-l)q_1}{1-\frac{l}{p_1}}\left[\frac{(1-d)^kp_1^k}{1-p_1}(1-(1-d)p_1) - \frac{q_1}{1-lp_1}(1-d)^kl^{k+1}p_1^{k-1}(1-(1-d)lp_1)\right] \\ &= \frac{(1-l)q_1}{p_1-l}\left[\frac{p_1(1-d)^kp_1^k}{q_1}(1-(1-d)p_1) - \frac{q_1p_1}{1-lp_1}(1-d)^kl^{k+1}p_1^{k-1}(1-(1-d)lp_1)\right]. \end{aligned} $$

Inserting the required formulae into A_x, we arrive at

$$ \begin{aligned} A_x &= \frac{(1-l)q_1}{p_1-l}\left[\frac{p_1}{q_1}(1-(1-d)p_1)\sum_{k=0}^{\infty}v^{k+1}(1-d)^kp_1^k\right. \\ &\quad \left. - \frac{q_1(1-(1-d)lp_1)}{1-lp_1}\sum_{k=0}^{\infty}v^{k+1}l^{k+1}(1-d)^kp_1^k\right] \\ &= \frac{(1-l)q_1}{p_1-l}\left[\frac{p_1}{q_1}(1-(1-d)p_1)\frac{v}{1-v(1-d)p_1} - \frac{q_1(1-(1-d)lp_1)}{1-lp_1} * \frac{vl}{1-v(1-d)lp_1}\right]. \end{aligned} $$

Similarily,

$$ A_y = \frac{(1-l)q_2}{p_2-l}\left[\frac{p_2}{q_2}(1-(1-d)p_2)\frac{v}{1-v(1-d)p_2} - \frac{q_2(1-(1-d)lp_2)}{1-lp_2} * \frac{vl}{1-v(1-d)lp_2}\right]. $$

When $d = 0$, it simplifies to A_x with only lengthening.

$$ A_x = \frac{(1-l)q_1}{p_1-l}\left[\frac{p_1}{q_1}(1-p_1)\frac{v}{1-vp_1} - \frac{q_1(1-lp_1)}{1-lp_1}\frac{vl}{1-vlp_1}\right]. $$

When $l = 0$, it simplifies to A_x with only the disaster.

$$ A_x = \frac{q_1}{p_1}\left[\frac{p_1}{q_1}(1-(1-d)p_1)\frac{v}{1-v(1-d)p_1}\right]. $$

Therefore,

$$ \begin{aligned} A_{\overline{xy}} &= \frac{(1-l)q_1}{p_1-l}\left[\frac{p_1}{q_1}(1-(1-d)p_1)\frac{v}{1-v(1-d)p_1} - \frac{q_1(1-(1-d)lp_1)}{1-lp_1}\frac{vl}{1-v(1-d)lp_1}\right] \\ &\quad + \frac{(1-l)q_2}{p_2-l}\left[\frac{p_2}{q_2}(1-(1-d)p_2)\frac{v}{1-v(1-d)p_2} - \frac{q_2(1-(1-d)lp_2)}{1-lp_2}\frac{vl}{1-v(1-d)lp_2}\right] \\ &\quad - \frac{1}{p_1p_2-l}\left[(1-l)(1-(1-d)p_1p_2)\frac{vp_1p_2}{1-v(1-d)p_1p_2} - (1-p_1p_2)(1-(1-d)l)\frac{vl}{1-v(1-d)l}\right]. \end{aligned} $$

Appendix A - Common Disaster

To verify that these probabilities are correct, we will check to see that they sum to one.

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} P(K_1^D = n_1, K_2^D = n_2) =$$

$$\sum_{n_2=0}^{\infty}\left(\sum_{n_1=0}^{n_2-1} P(K_1^D = n_1, K_2^D = n_2) + P(K_1^D = n_2, K_2^D = n_2) + \sum_{n_1=n_2+1}^{\infty} P(K_1^D = n_1, K_2^D = n_2)\right)$$

$$\begin{aligned}
&= \sum_{n_2=1}^{\infty}\sum_{n_1=0}^{n_2-1}[(1-d)^{n_2} p_1^{n_1} p_2^{n_2}(1-p_1)(1-(1-d)p_2)] \\
&\quad + \sum_{n_2=0}^{\infty}[(1-d)^{n_2} p_1^{n_1} p_2^{n_2}[1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2]] \\
&\quad + \sum_{n_2=0}^{\infty}\sum_{n_1=n_2+1}^{\infty}[(1-d)^{n_1} p_1^{n_1} p_2^{n_2}(1-p_2)(1-(1-d)p_1]] \\
&= (1-p_1)(1-(1-d)p_2)\sum_{n_2=1}^{\infty}[(1-d)^{n_2} p_2^{n_2}]\sum_{n_1=0}^{n_2-1} p_1^{n-1} \\
&\quad + \frac{1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2}{1-(1-d)p_1p_2} \\
&\quad + (1-p_2)(1-(1-d)p_1)\sum_{n_2=0}^{\infty} p_2^{n_2}\sum_{n_1=n_2+1}^{\infty}(1-d)^{n_1} p_1^{n_1}
\end{aligned}$$

Solve the inner summations

$$\begin{aligned}
&= (1-p_1)(1-(1-d)p_2)\sum_{n_2=1}^{\infty}(1-d)^{n_2} p_2^{n_2}\frac{1-p_1^{n_2}}{1-p_1} \\
&\quad + \frac{1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2}{1-(1-d)p_1p_2} \\
&\quad + (1-p_2)(1-(1-d)p_1)\sum_{n_2=0}^{\infty} p_2^{n_2}\frac{(1-d)^{n_2+1}p_1^{n_2+1}}{1-(1-d)p_1}
\end{aligned}$$

Solve the inner summations

$$\begin{aligned}
&= (1-(1-d)p_2)\sum_{n_2=1}^{\infty}[(1-d)^{n_2} p_2^{n_2} - (1-d)^{n_2} p_1^{n_2} p_2^{n_2}] \\
&\quad + \frac{1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2}{1-(1-d)p_1p_2} \\
&\quad + (1-p_2)(1-d)p_1\sum_{n_2=0}^{\infty}(1-d)^{n_2} p_1^{n_2} p_2^{n_2} \\
&= (1-(1-d)p_2)\left[\frac{(1-d)p_2}{1-(1-d)p_2} - \frac{(1-d)p_1p_2}{1-(1-d)p_1p_2}\right] \\
&\quad + \frac{1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2}{1-(1-d)p_1p_2} \\
&\quad + (1-d)p_1(1-p_2)\frac{1}{1-(1-d)p_1p_2}
\end{aligned}$$

Reduce over a common denominator

$$
\begin{aligned}
&= \frac{1}{1-(1-d)p_1p_2}[(1-d)p_2[1-(1-d)p_1p_2]-(1-d)p_1p_2[1-(1-d)p_2] \\
&\quad +1-(1-d)p_1-(1-d)p_2+(1-d)p_1p_2+(1-d)p_1(1-p_2)] \\
&= \frac{1}{1-(1-d)p_1p_2}[1-(1-d)p_1p_2] \\
&= 1.
\end{aligned}
$$

Appendix B - Life Lengthening

To verify that these probabilities are correct, check to see that they sum to one.

$$
\begin{aligned}
&\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} P(K_1^L = n_1, K_2^L = n_2) \\
&= \sum_{n_2=0}^{\infty}\left[\sum_{n_1=0}^{n_2} q_1q_2(1-l)\frac{p_1^{n_1+1}p_2^{n_2+1}-p_2^{n_2-n_1}l^{n_1+1}}{p_1p_2-l}\right. \\
&\quad \left. + \sum_{n_1=n_2+1}^{\infty} q_1q_2(1-l)\frac{p_1^{n_1+1}p_2^{n_2+1}-p_1^{n_1-n_2}l^{n_2+1}}{p_1p_2-l}\right] \\
&= \sum_{n_2=0}^{\infty}\left[\frac{q_1q_2(1-l)}{p_1p_2-l}\sum_{n_1=0}^{n_2} p_1^{n_1+1}p_2^{n_2+1}-p_2^{n_2-n_1}l^{n_2+1}\right. \\
&\quad \left. + \frac{q_1q_2(1-l)}{p_1p_2-l}\sum_{n_1=n_2+1}^{\infty} p_1^{n_1+1}p_2^{n_2+1}-p_1^{n_1-n_2}l^{n_2+1}\right]
\end{aligned}
$$

Solve the inner summations

$$
\begin{aligned}
&= \sum_{n_2=0}^{\infty}\left[\frac{q_1q_2(1-l)}{p_1p_2-l}\left(p_2^{n_2+1}\frac{p_1-p_1^{n_2+2}}{1-p_1}-\frac{p_2^{n_2}l-\frac{l^{n_2+2}}{p_2}}{1-\frac{l}{p_2}}\right)\right. \\
&\quad \left. + \frac{q_1q_2(1-l)}{p_1p_2-l}\left(\frac{p_1^{n_2+2}p_2^{n_2+1}}{1-p_1}-\frac{p_1l^{n_2+1}}{1-p_1}\right)\right]
\end{aligned}
$$

Reduce over a common denominator

$$
\begin{aligned}
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\frac{p_1p_2}{(1-p_1)(1-p_2)}-\sum_{n_2=0}^{\infty}\frac{p_2^{n_2+1}l-l^{n_2+2}}{p_2-l}-\frac{p_1l}{(1-p_1)(1-l)}\right] \\
&= \frac{q_1q_2(1-l)}{p_1p_2-l}\left[\frac{p_1p_2}{q_1q_2}-\frac{1}{p_2-l}\left(\frac{p_2l}{q_2}-\frac{l^2}{1-l}\right)-\frac{p_1l}{q_1(1-l)}\right]
\end{aligned}
$$

Simplify,

$$
\begin{aligned}
&= \frac{1}{p_1p_2-l}\left[p_1p_2(1-l)-\frac{1}{p_2-l}(q_1p_2l(1-l)-q_1q_2l^2)-p_1q_2l\right] \\
&= \frac{1}{p_1p_2-l}\left[p_1p_2(1-l)-p_1(1-p_2)l-\frac{(1-p_1)l}{p_2-l}(p_2(1-l)-(1-p_2)l)\right] \\
&= \frac{1}{p_1p_2-l}\left[p_1p_2-p_1l-\frac{(1-p_1l}{p_2-l}[p_2-p_2l-l+p_2l\right]
\end{aligned}
$$

Cancel terms

$$
\begin{aligned}
&= \frac{1}{p_1 p_2 - l} \left[p_1 p_2 - p_1 l - l + p_1 l \right] \\
&= \frac{1}{p_1 p_2 - l} \left[p_1 p_2 - l \right] \\
&= 1.
\end{aligned}
$$

Appendix C - Graphs

All graphs refer to values of $p_1, p_2 = 0.5$ and $v = 0.25$.

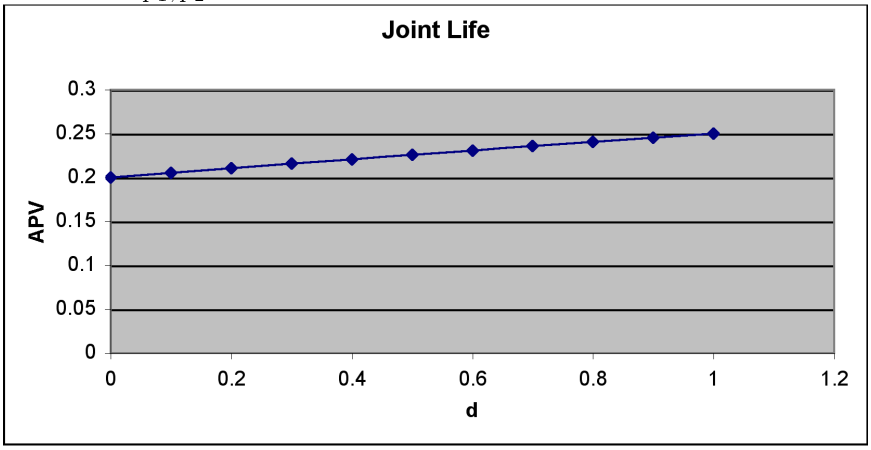

Figure 1: Graph of A_{xy} with common disaster.

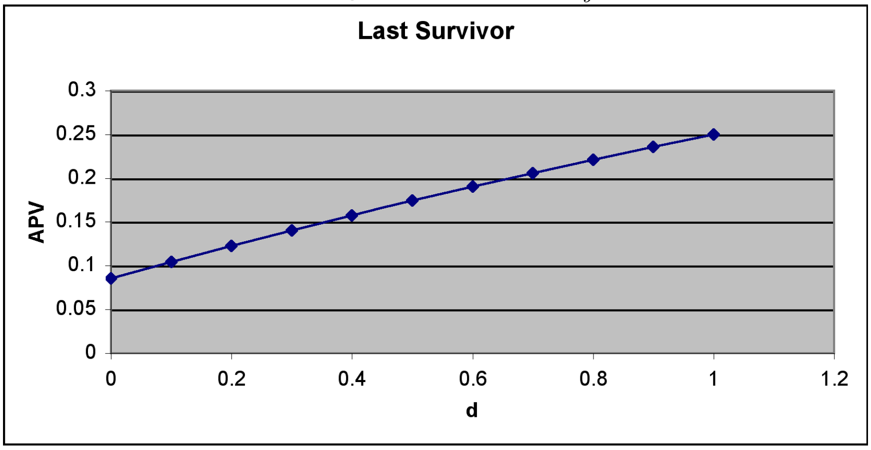

Figure 2: Graph of $A_{\overline{xy}}$ with common disaster.

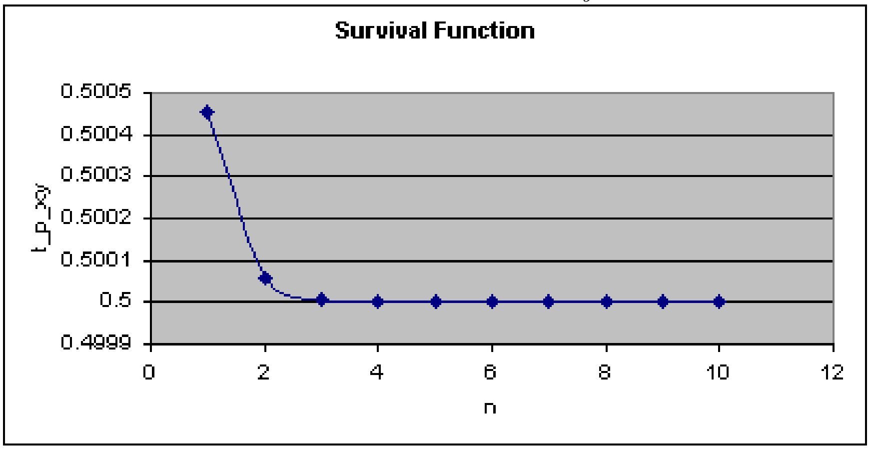

Figure 3: Graph of ${}_kp_{xy}$ with life lengthening, $k = 1$.

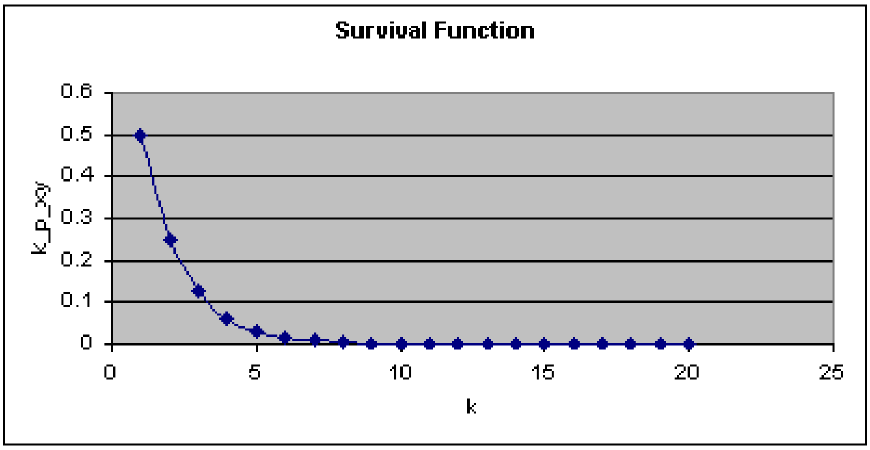

Figure 4: Graph of ${}_kp_{xy}$ with life lengthening, $n = 1$.

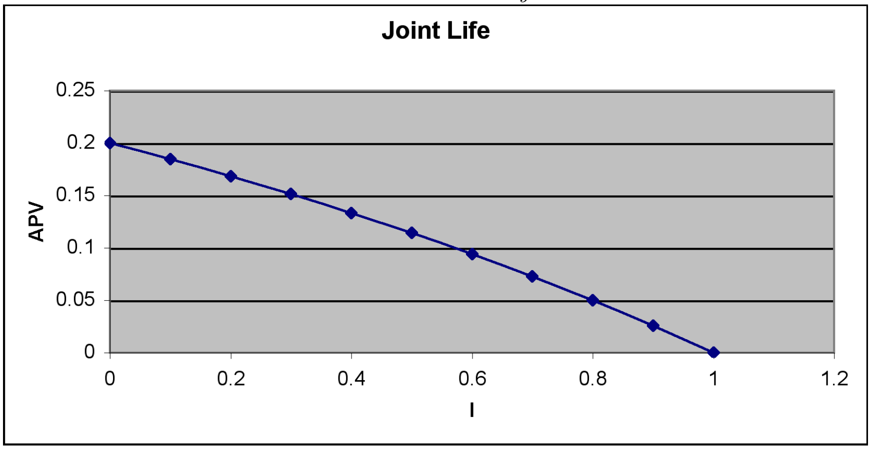

Figure 5: Graph of A_{xy} with life lengthening.

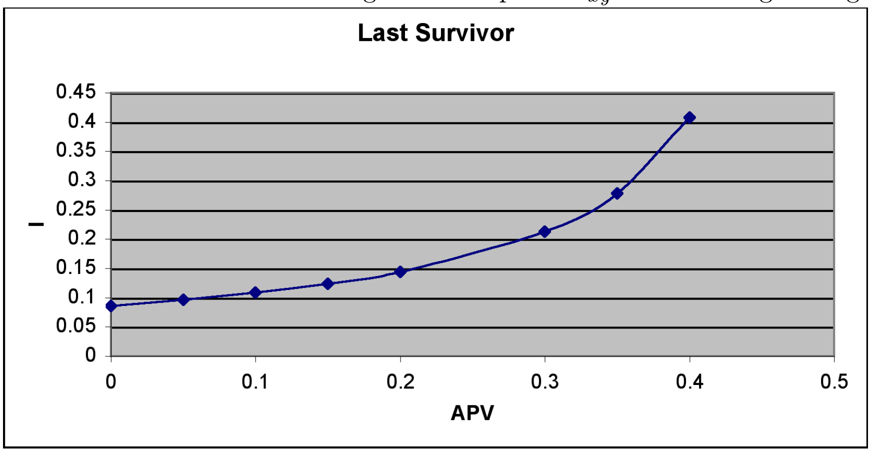

Figure 6: Graph of $A_{\overline{xy}}$ with life lengthening. For $l = .5$ and beyond the function is undefined due to the chosen values of $p_1, p_2,$ and v. $l = .25$ is also undefined but interpolated above.

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