The Time Value of Ruin in a Sparre Andersen Model:

Ruin Theory by Divided Differences

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1. Introduction

In a Sparre Andersen model, the claim counting process $\{N(t)\}$ is a renewal counting process. For j = 1, 2, 3, … , let T_j denote the time when the j-th claim of amount X_j occurs. Let

$$V_j = T_j - T_{j-1}, \qquad j = 2, 3, \ldots, \tag{1.1}$$

be the *interclaim time* random variables. The assumption is that $T_1, V_2, V_3, \ldots$ are positive i. i. d. random variables. We use V to denote a representative of these random variables. We assume that V has a probability density function and denote it as f_V.

We assume a constant premium rate c. The individual claims $\{X_j\}$ are positive i. i. d. random variables with probability density function p(x), and are independent of $\{N(t)\}$. The requirement of a positive *security loading* is

$$c\,\mathbf{E}[V] > \mathbf{E}[X], \tag{1.2}$$

where X denotes a representative of $\{X_j\}$.

A main goal in this paper is to evaluate the expectation

$$\phi(u) = \mathbf{E}[e^{-\delta T}\, w(U(T-), |U(T)|)\, 1_{(T<\infty)} \mid U(0) = u], \quad u \geq 0. \tag{1.3}$$

Here, the positive parameter δ can be interpreted as a force of interest, $U(t)$ is the surplus at time t,

$$U(t) = u + ct - \sum_{j=1}^{N(t)} X_j, \tag{1.4}$$

T is the *time of ruin*, $w(U(T-), |U(T)|)$ is the "penalty" at ruin, and $1_{(.)}$ is the indicator function. We shall obtain specific results under the assumption that V is the sum of a fixed number of independent, exponentially distributed random variables. This assumption will be used from Section 4 onwards. Note that the Erlang(n) model is the special case where these exponential random variables are identically distributed.

2. Renewal Equation for $\phi(u)$

We now determine a (defective) renewal equation for $\phi(u)$ by probabilistic reasoning. For $U(0) = u \geq 0$, let $f(x, y, t \mid u)$ denote the (defective) joint probability density function of $U(T-)$, $|U(T)|$, and T. Also, define

$$f(x, y \mid u) = \int_0^\infty e^{-\delta t} f(x, y, t \mid u)\, dt. \tag{2.1}$$

Consider the first time when the surplus falls below the initial level. The probability that this event occurs between time t and time t + dt, with

$$u + x \leq U(t-) \leq u + x + dx$$

and

$$u - y - dy \leq U(t) \leq u - y,$$

is

$$f(x, y, t \mid 0)\, dx\, dy\, dt.$$

Also, the occurrence y > u means that ruin takes place with this claim. Thus,

$$\phi(u) = \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t} \phi(u-y)\, f(x, y, t \mid 0)\, dt\, dx\, dy$$

$$+ \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x+u, y-u)\, f(x, y, t \mid 0)\, dt\, dx\, dy$$

$$= \int_0^u \int_0^\infty \phi(u-y)\, f(x, y \mid 0)\, dx\, dy$$

$$+ \int_u^\infty \int_0^\infty w(x+u, y-u)\, f(x, y \mid 0)\, dx\, dy, \tag{2.2}$$

which is the desired renewal equation.

It remains to determine $f(x, y \mid 0)$, the joint "discounted" probability density function of U(T–) and U(T) given that U(0) = 0. Setting u = 0 in (2.2), we have

$$\phi(0) = 0 + \int_0^\infty \int_0^\infty w(x, y)\, f(x, y \mid 0)\, dx\, dy. \tag{2.3}$$

If we can determine $\hat{\phi}$, the Laplace transform of ϕ, then we have another formula for $\phi(0)$ by means of the *initial value theorem* (Spiegel 1965, p. 5),

$$\phi(0) = \lim_{\xi \to \infty} \xi \hat{\phi}(\xi). \tag{2.4}$$

We shall see in Section 7 that, by comparing (2.3) with (2.4), we obtain an explicit formula for $f(x, y \mid 0)$, which is a key result in this paper.

With the definition

$$g(y) = \int_0^\infty f(x, y \mid 0)\, dx, \tag{2.5}$$

equation (2.2) can be written neatly as

$$\phi = \phi * g + h, \tag{2.6}$$

where $*$ denotes the convolution operation and the function h(u) is defined by the last integral in (2.2). The differential g(y)dy can be interpreted as the "discounted" probability that the surplus will ever fall below its initial value u, and will be between u – y and u – y – dy when it happens for the first time.

3. Lundberg's Fundamental Equation

For k = 1, 2, 3, … , let U_k denote the surplus immediately after the payment of the k-th claim,

$$U_k = u + cT_k - \sum_{j=1}^{k} X_j. \tag{3.1}$$

We seek numbers ξ for which the sequence of random variables,

$$\{ e^{-\delta T_k + \xi U_k};\ k = 0, 1, 2, \dots \}, \tag{3.2}$$

becomes a *martingale*. In (3.2), the term for k = 0 is the constant $e^{\xi u}$. The martingale condition is

$$\hat{f}_V(\delta - c\xi)\, \hat{p}(\xi) = 1, \tag{3.3}$$

where $\hat{f}_V$ and $\hat{p}$ are the *Laplace transforms* of the probability density functions f_V and p, respectively. Equation (3.3) is a generalization of *Lundberg's fundamental equation.* Its solutions in the right half of the complex plane play an important role in this paper.

Suppose that equation (3.3) has a negative solution –R. (R can be called the *adjustment coefficient.*) Then, with ξ = –R, the sequence (3.2) is a bounded martingale before ruin. By applying the *optional sampling theorem*, we see that, with $w(x, y) = e^{Ry}$,

$$\phi(u) = e^{-Ru}. \tag{3.4}$$

4. A Specific Assumption on Interclaim Time

In the rest of this paper, we assume that V is the sum of n independent, exponentially distributed random variables, say with means $1/\lambda_1$, $1/\lambda_2$, … , $1/\lambda_n$. Thus,

$$\mathbf{E}[e^{-\xi V}] = \hat{f}_V(\xi) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \xi}. \tag{4.1}$$

Let $\gamma(\xi)$ be the reciprocal of $\hat{f}_V(\delta - c\xi)$. Then, equation (3.3) becomes

$$\gamma(\xi) - \hat{p}(\xi) = 0. \tag{4.2}$$

By (4.1),

$$\gamma(\xi) = \prod_{j=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j}\xi\right], \tag{4.3}$$

which is an n-th degree polynomial.

Let D denote the *differentiation operator*. Gerber and Shiu (2003, 2005) have shown that the function ϕ satisfies the integro-differential equation

$$\gamma(D)\phi = \phi * p + \omega, \tag{4.4}$$

where

$$\omega(u) = \int_0^{\infty} w(u, y)\, p(u + y)\, dy. \tag{4.5}$$

We shall solve (4.4) in terms of *Laplace transforms* in Section 6. In preparation, we now examine the solutions of equation (4.2) in the right half of the complex plane.

We claim that, in the right half of the complex plane, equation (4.2) has exactly n solutions. To see this, consider a domain that is a half disk centered at 0, lying in the right half of the complex plane, and with a sufficiently large radius. For Re $\xi \geq 0$, we have $|\hat{p}(\xi)| \leq 1$. Because $\gamma(\xi)$ has exactly n zeros and they are positive, our claim follows from *Rouché's theorem* if we can show that $|\gamma(\xi)| > 1$ on the boundary of such a half disk. It is obvious from (4.3) that $|\gamma(\xi)| > 1$ for $|\xi|$ sufficiently large. Now, for $\text{Re}\,\xi = 0$, ξ lying on the imaginary axis, we have

$$|\gamma(\xi)| \ \geq \ \prod_{j=1}^{n} \frac{\lambda_j + \delta}{\lambda_j} \ > \ 1$$

also. We denote these n roots of (4.2) in the right half of the complex plane as $\rho_1, \rho_2, \ldots, \rho_n$.

5. Divided Differences

This section presents a brief review on *divided differences* (Freeman 1960; Steffensen 1950). For a function $\eta(s)$, its divided differences, with respect to distinct numbers $r_1, r_2, r_3, \ldots$, can be defined recursively as follows:

$$\eta(s) \ = \ \eta(r_1) \ + \ (s - r_1)\eta[r_1, s],$$

$$\eta[r_1, s] \ = \ \eta[r_1, r_2] \ + \ (s - r_2)\eta[r_1, r_2, s],$$

$$\eta[r_1, r_2, s] \ = \ \eta[r_1, r_2, r_3] \ + \ (s - r_3)\eta[r_1, r_2, r_3, s],$$

and so on. Hence, if the function h(s) vanishes at $s = r_1, r_2, \ldots, r_n$, then

$$\eta(s) \ = \ \eta[s, r_1, r_2, \ldots, r_n] \prod_{k=1}^{n}(s - r_k). \qquad (5.1)$$

Also, we have the following formula for the (k–1)-th divided difference

$$\eta[r_1, r_2, \ldots, r_k] \ = \ \sum_{j=1}^{k} \frac{\eta(r_j)}{\prod_{i=1, i \neq j}^{k}(r_j - r_i)}. \qquad (5.2)$$

It is useful to note that a divided difference with repeated *points of collocation* can be evaluated as a derivative. For example, if a, b and c are three distinct numbers, then

$$\eta[a, a, a, b, b, c] = \frac{1}{(3-1)!}\frac{1}{(2-1)!}\frac{\partial^2}{\partial a^2}\frac{\partial}{\partial b}\eta[a, b, c], \tag{5.3}$$

a contour-integration proof of which can be found in Shiu (1983). Because of this property of divided differences, in the rest of this paper we shall assume that the roots of equation (4.2) in the right half of the complex plane, $\rho_1, \rho_2, \dots, \rho_n$, are distinct.

6. The Laplace Transform of ϕ

We now solve the integro-differential equation (4.4) in terms of Laplace transforms. Let f(u) be a function with Laplace transform $\hat{f}(\xi)$; the Laplace transform of the k-th derivative $f^{(k)}(u)$ is

$$\xi^k\hat{f}(\xi) - \xi^{k-1}f(0) - \xi^{k-2}f'(0) - \dots - f^{(k-1)}(0)$$

(Spiegel 1965, p. 10). Thus, the Laplace transform of (4.4) is

$$\gamma(\xi)\hat{\phi}(\xi) + q(\xi) = \hat{\phi}(\xi)\hat{p}(\xi) + \hat{\omega}(\xi), \qquad \text{Re } \xi \ge 0, \tag{6.1}$$

where $q(\xi)$ is a polynomial of degree n–1 or less, with coefficients in terms of δ, c, $\lambda_1, \lambda_2, \dots, \lambda_n$, and the values of $\phi(u)$ and its first n–1 derivatives at u = 0. It follows from (6.1) that

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\gamma(\xi) - \hat{p}(\xi)}, \qquad \text{Re } \xi \ge 0. \tag{6.2}$$

Because $\hat{\phi}(\xi)$ is finite for Re $\xi \ge 0$, the numerator on the right-hand side of (6.2) must be zero whenever the denominator is zero. We have shown in Section 4 that, in the right half of the complex plane, the function in the denominator of (6.2) has n zeros $\rho_1, \rho_2, \dots, \rho_n$. We apply (5.1) to the numerator and to the denominator of (6.2), with $s = \xi$ and $r_k = \rho_k$. After canceling $\prod_{k=1}^{n}(\xi - \rho_k)$, we obtain

$$\hat{\phi}(\xi) = \frac{\hat{\omega}[\xi,\rho_1,\ldots,\rho_n] - q[\xi,\rho_1,\ldots,\rho_n]}{\gamma[\xi,\rho_1,\ldots,\rho_n] - \hat{p}[\xi,\rho_1,\ldots,\rho_n]}, \qquad \text{Re}\ \xi \geq 0. \tag{6.3}$$

Because $q(\xi)$ is a polynomial of degree n−1 or less, we have $q[\xi, \rho_1, \rho_2, \ldots, \rho_n] = 0$.

Because $\gamma(\xi)$ is a polynomial of degree n, the n-th divided difference $\gamma[\xi, \rho_1, \rho_2, \ldots, \rho_n]$ is the coefficient of ξ^n in $\gamma(\xi)$; we shall denote this leading coefficient as γ_n. Thus, equation (6.3) simplifies as

$$\hat{\phi}(\xi) = \frac{\hat{\omega}[\xi,\rho_1,\ldots,\rho_n]}{\gamma_n - \hat{p}[\xi,\rho_1,\ldots,\rho_n]}, \qquad \text{Re}\ \xi \geq 0. \tag{6.4}$$

It is easy to see from (4.3) that

$$\gamma_n = \prod_{j=1}^{n} \frac{-c}{\lambda_j}. \tag{6.5}$$

In some cases, the function ϕ can be determined by identifying the right-hand side of (6.4). In general, we have the *inversion formula*:

$$\phi(u) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \hat{\phi}(\xi)\, e^{\xi u}\, d\xi, \tag{6.6}$$

where $i = \sqrt{-1}$, and the path of integration is parallel to the imaginary axis in the complex plane, with the real number b being chosen so that all the singularities of the integrand lie to the left of the line of integration (Spiegel 1965, p. 201). However, the integral can be difficult to calculate, unless $\hat{\phi}$ is a rational function.

7. A Key Result

It follows from (2.4) and (6.4) that

$$\phi(0) = \lim_{\xi\to\infty} \xi \frac{\hat{\omega}[\xi,\rho_1,\ldots,\rho_n]}{\gamma_n - \hat{p}[\xi,\rho_1,\ldots,\rho_n]}. \tag{7.1}$$

Now,

$$\lim_{\xi\to\infty} \hat{p}[\xi, \rho_1, \rho_2, \dots, \rho_n] = 0$$

because $\hat{p}(\xi) \to 0$, and

$$\lim_{\xi\to\infty} \xi\,\hat{\omega}[\xi, \rho_1, \rho_2, \dots, \rho_n] = \lim_{\xi\to\infty} \xi\,\frac{\hat{\omega}[\xi,\rho_2,\dots,\rho_n] - \hat{\omega}[\rho_1,\rho_2,\dots,\rho_n]}{\xi - \rho_1}$$

$$= 0 - \hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]. \tag{7.2}$$

Thus, (7.1) becomes

$$\phi(0) = \frac{-\hat{\omega}[\rho_1,\rho_2,\dots,\rho_n]}{\gamma_n} = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}(-1)^{n-1}\hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]. \tag{7.3}$$

It follows from (4.5) that the Laplace transform of ω is

$$\hat{\omega}(\rho) = \int_0^\infty\int_0^\infty e^{-\rho x}\, w(x, y)\, p(x + y)\, dx\, dy. \tag{7.4}$$

With the definition $\varepsilon_x(\rho) = e^{-\rho x}$, we have

$$\hat{\omega}[\rho_1, \rho_2, \dots, \rho_n] = \int_0^\infty\int_0^\infty \varepsilon_x[\rho_1, \rho_2, \dots, \rho_n]\, w(x, y)\, p(x + y)\, dx\, dy. \tag{7.5}$$

Substituting (7.5) in (7.3) and then comparing with (2.3), we obtain a key result in this paper:

$$f(x, y\,|\,0) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\, p(x + y)(-1)^{n-1}\varepsilon_x[\rho_1, \rho_2, \dots, \rho_n]. \tag{7.6}$$

For n = 1, this result is implicitly contained in (2.33) and explicitly given in (3.3) of Gerber and Shiu (1998).

8. Li's Renewal Equation

The function g in the (defective) renewal equation (2.6) is defined by (2.5). Applying (7.6) to (2.5) yields

$$g(y) = (-1)^{n-1}\frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\int_0^\infty p(x+y)\,\varepsilon_x[\rho_1, \rho_2, \ldots, \rho_n]\,dx. \tag{8.1}$$

Thus, we are interested in integrals of the form

$$\int_0^\infty p(x+y)\,e^{-\rho x}\,dx, \tag{8.2}$$

which can be viewed as the Laplace transform, for argument ρ, of the translated function $p(y + \bullet)$. Following Dickson and Hipp (2001, p. 336), we consider the linear operator T_ρ defined as follows. For a number ρ with nonnegative real part, $\mathrm{Re}\,\rho \geq 0$, and for an integrable function φ,

$$(T_\rho\varphi)(y) = \int_0^\infty e^{-\rho x}\,\varphi(y+x)\,dx, \qquad y \geq 0. \tag{8.3}$$

Furthermore, we define

$$S = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\prod_{j=1}^{n} T_{\rho_j}. \tag{8.4}$$

We shall see that the operators T's commute,

$$T_\rho\,T_\xi = T_\xi\,T_\rho, \tag{8.5}$$

and hence, the product of operators, $\prod_{j=1}^{n} T_{\rho_j}$, is unambiguous. A main goal in this section is to show that equation (8.1) is

$$g = Sp \tag{8.6}$$

and that

$$h = S\omega. \tag{8.7}$$

In other words, the renewal equation (2.6) can be written as

$$\phi = \phi*(Sp) + S\omega. \tag{8.8}$$

With all λ's being identical, the renewal equation (8.8) was first given by Li (2003, Theorem 2). See also Li and Garrido (2004). We shall call (8.8) *Li's Renewal Equation.* For n = 1, (8.8) is (2.34) of Gerber and Shiu (1998).

To understand of the operator T_ρ, recall the *translation operator* (*shift operator*) E that actuaries used to learn in *finite differences*. Since $E^x\varphi(y) = \varphi(y + x)$, we have the operator equation

$$T_\rho = \int_0^\infty e^{-\rho x} E^x \, dx. \tag{8.9}$$

Actuarial authors such as Steffensen (1950, p. 186) and Freeman (1960, p. 127) have noted that Taylor's formula leads to the operator identity

$$E^x = e^{xD}, \tag{8.10}$$

with which (8.9) becomes

$$T_\rho = \int_0^\infty e^{-x(\rho I - D)} \, dx = (\rho I - D)^{-1}. \tag{8.11}$$

That such operators commute with each other, i.e., that equation (8.5) holds, is an immediate consequence of (8.11).

A rigorous discussion of (8.11) can be found in Section 1.3.3 of Butzer and Berens (1967). It is shown that the operator equation is valid under the assumption that Re $\rho > 0$ and that the domain of the operators is the Banach space of bounded, uniformly continuous functions on $[0, \infty)$. As an operator-valued function of ρ, T_ρ is called the *resolvent* of the operator D. Because the differentiation operator D is an unbounded operator, some authors would call T_ρ a *pseudo-resolvent* (Hille and Phillips 1957).

By (8.11), the partial fraction formula

$$\prod_{j=1}^{n} \frac{1}{\rho_j - z} = \sum_{j=1}^{n} \Big(\prod_{i=1, i\neq j}^{n} \frac{1}{\rho_i - \rho_j} \Big) \frac{1}{\rho_j - z} \tag{8.12}$$

can be translated into the operator identity

$$\prod_{j=1}^{n} T_{\rho_j} = \sum_{j=1}^{n} \Big(\prod_{i=1, i\neq j}^{n} \frac{1}{\rho_i - \rho_j} \Big) T_{\rho_j}. \tag{8.13}$$

We can now show that (8.1) is the same as (8.6). The divided difference in the integrand of (8.1) can be expanded using formula (5.2), with $k = n$, $\eta(.) = \varepsilon_x(.)$ and $r = \rho$. Then the integral in (8.1) becomes a linear combination of n integrals of the form (8.2). Note that the denominator in (5.2) is of the form $r_j - r_i$, while the denominator in (8.13) is of the form $\rho_i - \rho_j$; this difference accounts for the factor of $(-1)^{n-1}$ in (8.1). We apply (8.13) and (8.4) to conclude that the right-hand of (8.1) is (Sp)(y). This proves equation (8.6).

Similarly, we can derive (8.7). Hence, we have proved Li's renewal equation (8.8).

Remark Li's renewal equation (8.8) follows from the integro-differential equation (4.4), if it can shown that, for each integrable function φ,

$$S[\gamma(D)\varphi - \varphi * p] = \varphi - \varphi*(Sp).$$

This is the approach given by Gerber and Shiu (2003, 2005), who show inductively that, if $S_k = \prod_{j=1}^{k} T_{\rho_j}$, then

$$S_k[\gamma(D)\varphi - \varphi * p] = (-1)^k \gamma[\rho_1, \rho_2, \dots, \rho_k, D]\varphi - \varphi*(S_k p), \quad k = 1, 2, ,,, , n.$$

Note that $\gamma[\rho_1, \rho_2, \dots, \rho_n, D] = \gamma_n I$, and the coefficient γ_n is given by (6.5).

9. The Laplace Transform of g

As a check, let us take the Laplace transform of (8.8) and derive (6.4). Observe that, for an integrable function φ,

$$\widehat{\varphi}(\xi) = (T_\xi \varphi)(0), \qquad \mathrm{Re}\,\xi \geq 0. \tag{9.1}$$

Thus, taking the Laplace transform of (8.8) and applying (9.1) yields

$$\hat{\phi}(\xi) = \frac{(T_\xi(S\omega))(0)}{1-(T_\xi(Sp))(0)}. \tag{9.2}$$

Now, it follows from (8.13), (9.1) and (5.2) that

$$[(\prod_{j=1}^{m} T_{r_j})\varphi](0) = (-1)^{m-1}\widehat{\varphi}[r_1, r_2, \dots, r_m]. \tag{9.3}$$

Hence,

$$(T_\xi(Sp))(0) = ((T_\xi S)p)(0) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}(-1)^n\widehat{p}[\xi, \rho_1, \rho_2, \dots, \rho_n].$$

(9.4)

and

$$(T_\xi(S\omega))(0) = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}(-1)^n\widehat{\omega}[\xi, \rho_1, \rho_2, \dots, \rho_n]. \tag{9.5}$$

Therefore, (6.4) follows from (9.2), (9.4), (9.5) and (6.5).

We now derive an alternative formula for $\hat{g}(\xi)$. It follows from the identity $\widehat{p} = \gamma + (\widehat{p} - \gamma)$ that

$$\widehat{p}[\xi, \rho_1, \dots, \rho_n] = \gamma[\xi, \rho_1, \dots, \rho_n] + (\widehat{p} - \gamma)[\xi, \rho_1, \dots, \rho_n]. \tag{9.6}$$

The first term on the right-hand side of (9.6) is the coefficient γ_n, which is given by (6.5), while the second term can be evaluated using (5.1). Thus,

$$\hat{p}[\xi, \rho_1, \ldots, \rho_n] = \prod_{j=1}^{n} \frac{-c}{\lambda_j} + [\hat{p}(\xi) - \gamma(\xi)] \prod_{k=1}^{n} \frac{1}{\xi - \rho_k}, \tag{9.7}$$

applying which to (9.4) yields

$$\hat{g}(\xi) = (T_\xi Sp)(0) = 1 + \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n (\rho_1 - \xi)(\rho_2 - \xi) \cdots (\rho_n - \xi)} [\hat{p}(\xi) - \gamma(\xi)]. \tag{9.8}$$

Two interesting results immediately follow from formula (9.8). The first result is that, for $\delta > 0$,

$$\begin{aligned}
\mathbf{E}[e^{-\delta T} 1_{(T<\infty)} \mid U(0) = 0] &= \int_0^\infty g(y)\, dy \\
&= \hat{g}(0) \\
&= 1 + \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n \rho_1 \rho_2 \cdots \rho_n} [1 - \prod_{j=1}^{n} (1 + \frac{\delta}{\lambda_j})] \\
&= 1 - \frac{\prod_{k=1}^{n} (\lambda_k + \delta) - \lambda_1 \lambda_2 \cdots \lambda_n}{c^n \rho_1 \rho_2 \cdots \rho_n},
\end{aligned} \tag{9.9}$$

which is formula (10) in Li (2003). For $n = 1$, this is formula (3.9) of Gerber and Shiu (1998). For $n = 2$, it is the last formula in Dickson and Hipp (2001). The last part of Theorem 4 in Li and Garrido (2004) evaluates the limit $\delta \downarrow 0$ in the Erlang(n) case.

The second result following from (9.8) is that to solve for a negative ξ satisfying the equation $\hat{g}(\xi) = 1$ is equivalent to solve for a negative ξ satisfying (4.2). For $n = 1$, this equivalence has been pointed out in Remark (v) on page 54 of Gerber and Shiu (1998). Such a ξ, the negative of which is the adjustment coefficient R, is needed for obtaining an *asymptotic formula* for $\phi(u)$; see Section 4 of Gerber and Shiu (1998).

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References

Butzer, Paul L., and Hubert Berens. 1967. *Semi-Groups of Operators and Approximation.* New York: Springer-Verlag.

Dickson, David C.M., and Christian Hipp. 2001. "On the Time to Ruin for Erlang(2) Risk Processes," *Insurance: Mathematics and Economics* 29: 333-44.

Freeman, Harry. 1960. *Finite Differences for Actuarial Students*. Cambridge, U.K.: Cambridge University Press.

Gerber, Hans U., and Elias S.W. Shiu. 1998. "On the Time Value of Ruin," *North American Actuarial Journal* 2(1): 48-72; Discussions: 72-78.

Gerber, Hans U., and Elias S.W. Shiu. 2003. Discussion of Yebin Cheng and Qihe Tang, "Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process," *North American Actuarial Journal* 7(3): 117-19; 7(4): 96-101.

Gerber, Hans U., and Elias S.W. Shiu. 2005. "On the Time Value of Ruin in a Sparre Andersen Model," *North American Actuarial Journal* 9 (2), to appear.

Hille, Einar, and Ralph S. Phillips. 1957. *Functional Analysis and Semi-Group.* Revised Edition. Providence, RI: American Mathematical Society.

Li, Shuanming. 2003. Discussion of Yebin Cheng and Qihe Tang, “Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process,” *North American Actuarial Journal* 7(3): 119-22.

Li, Shuanming, and José Garrido. 2004. “On Ruin for the Erlang(n) Risk Process,” *Insurance: Mathematics and Economics* 34: 391-408.

Shiu, Elias S.W. 1983. “Divided Differences by Contour Integration,” *Actuarial Research Clearing House* 1983.1: 13-19.

Spiegel, Murray R. 1965. *Schaum’s Outline of Theory and Problems of Laplace Transforms.* New York: Schaum.

Steffensen, J.F. 1950. *Interpolation,* 2nd Edition. New York: Chelsea.