Analysis of the ruin probability using Laplace transforms and Karamata Tauberian theorems

Corina Constantinescu *and Enrique Thomann †

Abstract

The classical result of Cramer-Lundberg states that if the rate of premium, c, exceeds the average of the claims paid per unit time, $\lambda\mu$, then the probability of ruin of an insurance company decays exponentially fast as the initial capital $u \to \infty$. In this note, the asymptotic behavior of the probability of ruin is derived by means of infinitesimal generators and Laplace transforms. Using these same tools, it is shown that the probability of ruin has an algebraic decay rate if the insurance company invests its capital in a risky asset with a price which follows a geometric Brownian motion. The latter result is shown to be valid not only for exponentially distributed claim amounts, as in Frolova et al. (2002), but, more generally, for any claim amount distribution that has a moment generating function defined in a neighborhood of the origin.

1 Introduction

The collective risk model, introduced by Cramer and Lundberg in 1930, remains the subject of analysis by actuaries and mathematicians. In particular, different methods of analysis continue to be used in obtaining bounds or asymptotics of the ruin probability, under specific conditions on the claim size distribution (Cramer, 1930; Gerber, 1973; Ross, 1996). For example, in the classical risk model under the Cramer-Lundberg condition, the ruin probability presents an exponential decay as the initial capital $u \to \infty$ (Cramer, 1930). If the Cramer-Lundberg condition is weakened, the asymptotic behavior of the ruin probability changes dramatically. For instance, in the

*Oregon State University, Department of Mathematics, e-mail:corina@math.orst.edu
†Oregon State University, Department of Mathematics, e-mail:thomann@math.orst.edu

case of sub-exponentially distributed claim sizes, the ruin is asymptotically determined by a large claim (Embrechts et al., 1997).

Some recent studies impose a perturbation on the classical model. These perturbations model uncertainties in the rate at which premiums are being collected, or the rate of returns on the investment strategies by the insurance company. The perturbations may influence the asymptotic behavior of the ruin probability. For example, if the perturbation is a Brownian motion, the ruin probability still presents an exponential decay (Schmidli, 1995). On the other hand, if the risk model is altered by a geometric Brownian motion, then the asymptotic decay rate of the ruin probability is at best algebraic. This latter perturbation models the risk when the company invests its capital into a risky asset with returns following an exponential of a Brownian motion (Frolova et al., 2002).

In this paper, the asymptotics of the ruin probability are derived by analyzing the properties of the Laplace transform of the ruin probability. Using this technique, the exponential decay of the classical Cramer Lundberg model has a simple derivation. The same method is used in extending the cited results of Frolova et al. (2002) to more general claim size distributions than those considered in the original work. Specifically, while the result in Frolova et al. (2002) holds for claim sizes with an exponential distribution, the result presented in this paper applies to claim sizes that have a moment generating function defined in a neighborhood of the origin. Furthermore, since this approach relies on the Karamata Tauberian theorems, it appears to be easier to use under different assumptions on claim inter-arrival times and investment strategies.

We proceed in the following manner. In the next section, we consider the classical ruin problem and obtain the asymptotic behavior of the ruin probability. The integro-differential equation satisfied by the ruin probability is solved using elementary properties of the Laplace transform. In Section 3 we investigate the asymptotic behavior of the ruin probability under uncertain investments. The analysis starts once again using the infinitesimal generator of the risk process and the Laplace transform. However, given the more intricate form of the equation, we rely on the Karamata Tauberian theorem to obtain the asymptotic behavior. Some conclusions and research objectives are presented in Section 4. Finally, we include an Appendix with a summary of some technical tools used in this paper.

2 The classical Cramer Lundberg model

The classical risk model

$$X_t = u + ct - \sum_{k=1}^{N(t)} \xi_k,$$

is a compound Poisson process. X_t describes the evolution of the capital of an insurance company, that charges premiums at a constant rate c. The number of claims up to time $t \geq 0$ is a Poisson process $N(t)$, with a Poisson rate λ. The size or amount of the k-th claim is a random variable ξ_k. The claim sizes $\xi_1, \xi_2, ..$ are independent, identically distributed random variables, having the distribution function F, with positive mean μ and finite variance. The claims up to time t occur at random times $t_1, t_2, ..t_{N(t)}$. $Y_1 = t_1$, $Y_k = t_k - t_{k-1}$, for $k = 2, 3..$, are independent exponentially distributed random variables with finite mean λ. $(\xi_k)_k$ and $(Y_k)_k$ are independent.

The ruin probability in finite time is defined as $\Psi(u, T) = P(X_t < 0$, for some $t \leq T)$, where $0 < T < \infty$ and $u \geq 0$. The ruin probability

$$\Psi(u) = \Psi(u, \infty),$$

refers to the ruin probability with infinite horizon. The purpose of this paper is to study the asymptotic decay of the ruin probability as the initial capital u tends to infinity.

The notation for the Laplace transform of the ruin probability is $\hat{\Psi}(s) = \mathcal{L}\Psi(u)(s)$ and for the Laplace transform of the density of the claim sizes is $\mathcal{F}(s) = \mathcal{L}(f(\xi))(s)$. $M_\xi(-s)$ denotes the moment generating function of the distribution of the claim amounts ξ, and obviously, $\mathcal{F}(s) = M_\xi(-s)$.

Theorem 1. *Consider the Cramer-Lundberg model*

$$X_t = u + ct - \sum_{k=1}^{N(t)} \xi_k,$$

under the net profit condition $\lambda\mu < c$. Assume that $R > 0$ is the smallest positive number such that $-R$ is a solution of the Lundberg equation

$$cs - \lambda(1 - \mathcal{F}(s)) = 0.$$

Denote $\hat{\Psi}(s)(s + R) - \Psi(0) := \hat{H}(s)$. If $\hat{H}$ is the Laplace transform of a function H, then the limit $\lim_{u\to\infty} \Psi(u)e^{Ru}$ exists and moreover,

$$\lim_{u\to\infty} \Psi(u)e^{Ru} = \frac{\lambda\mu}{c + \lambda\mathcal{F}'(-R)} = \frac{\lambda\mu - c}{c - \lambda M'_\xi(R)}.$$

Proof. Using classical renewal theory or computing the infinitesimal generator of the risk model $A\Psi(u) = 0$, as in Paulsen and Gjessing (1997) (see Appendix), one can show that the ruin probability satisfies

$$c\Psi'(u) + \lambda \int_0^u \Psi(u-y)dF(y) - \lambda\Psi(u)(1 - F(u)) = 0, \tag{1}$$

with $\Psi(0) = \frac{\lambda\mu}{c}$. The Laplace transform of this equation,

$$s\hat{\Psi}(s) - \Psi(0) + \frac{\lambda}{c}\hat{\Psi}(s)\mathcal{F}(s) - \frac{\lambda}{c}\hat{\Psi}(s) + \frac{\lambda}{c}(\frac{1}{s} - \frac{\mathcal{F}(s)}{s}) = 0$$

has the solution

$$\hat{\Psi}(s) = \frac{c\Psi(0) - \frac{\lambda}{s}(1 - \mathcal{F}(s))}{cs - \lambda(1 - \mathcal{F}(s))}.$$

Recall that the denominator in this expression, $L(s) = cs - \lambda(1 - \mathcal{F}(s))$ gives the Lundberg equation $L(s) = 0$. Since the numerator doesn't vanish at $-R$, $-R$ is a pole of $\hat{\Psi}(s)$. By hypothesis,

$$\hat{\Psi}(s) = \frac{1}{s+R}\hat{H}(s) + \frac{1}{s+R}\Psi(0).$$

As the Laplace transform of a product of two functions is the Laplace transform of the convolution of the given functions, by the uniqueness of the Laplace transform it follows that

$$\Psi(u) = \int_0^u e^{-(u-t)R} H(t)dt + e^{-Ru}\Psi(0). \tag{2}$$

Since $\hat{H}(-R)$ is defined, passing to limit one has

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = \int_0^\infty e^{-(-R)t}H(t)dt + \Psi(0) = \hat{H}(-R) + \Psi(0).$$

Since $L(-R) = 0$,

$$\hat{H}(s) + \Psi(0) = \hat{\Psi}(s)(s+R) = \frac{c\Psi(0) - \frac{\lambda}{s}(1 - \mathcal{F}(s))}{\frac{L(S) - L(-R)}{s+R}}$$

and $(1 - \mathcal{F}(-R)) = \frac{c}{\lambda}(-R)$. Therefore, as $s \to -R$,

$$\hat{H}(-R) + \Psi(0) = \frac{c\Psi(0) - \frac{\lambda}{-R}\frac{c}{\lambda}(-R)}{L'(-R)} = \frac{c\Psi(0) - \frac{\lambda}{-R}\frac{c}{\lambda}(-R)}{c + \lambda\mathcal{F}'(-R)}.$$

The initial condition, $\Psi(0) = \frac{\lambda\mu}{c}$, together with $\mathcal{F}'(-R) = -M'_\xi(R)$, imply

$$\lim_{u\to\infty} e^{-Ru}\Psi(u) = \hat{H}(-R) + \Psi(0) = \frac{\lambda\mu - c}{c + \lambda\mathcal{F}'(-R)} = \frac{\lambda\mu - c}{c - \lambda M'_\xi(R)}, \qquad (3)$$

in agreement with Rolski et al. (1999). □

Remark 1. *If the claim sizes are exponentially distributed, the conditions on the Laplace transform of the ruin probability are trivially satisfied, since* $\hat{H}(s) = 0$. *In general, the existence of the function* H *depends upon the tail of the distribution of the claim sizes.*

3 The Cramer Lundberg model with investments

This section identifies the effects of a risky investment on the asymptotic behavior of the probability of ruin. If the insurance company invests the capital in an asset with a price that follows a geometric Brownian motion, with drift a and volatility σ, then the ruin probability has an algebraic decay rate or equals one, depending only on the parameters a and σ of the asset (Frolova et al., 2002). In the cited paper the result is established only for exponentially distributed claim sizes, because the method of proof relies on special properties of the exponential functions. A generalization of the result for distributions of the claim sizes having moment generating functions defined on a neighborhood of the origin is possible.

When the company capital is invested in a risky asset, the risk process is given by

$$X_t = u + a\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dW_s + ct - \sum_{k=0}^{N(t)} \xi_k, \qquad (4)$$

where ξ_k represents the size of the k-th claim, with the probability distribution function F on $(0,\infty)$, c is the fixed rate of premium and u is the initial capital. The capital X_t is continuously invested in a risky asset, with relative price increments $dX_t = adt + \sigma dW_t$, where a and σ are the drift and volatility of the returns of the asset.

Theorem 2. *Consider the model given by (4) and assume that* $\sigma > 0$. *Assume also that the distribution of the claims sizes* F *has a moment generating function defined on a neighborhood of the origin. Then:*

- *If the ruin probability decays at infinity, then*

$$2a/\sigma^2 > 1.$$

- *If* $1 < 2a/\sigma^2 < 2$, *then for some* $K > 0$,

$$\lim_{u\to\infty} \Psi(u)u^{2a/\sigma^2-1} = K.$$

Proof. Let $\rho = 2a/\sigma^2$. Consider the function

$$U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ \int_0^u \Psi(x)dx & \text{if} \quad u \geq 0. \end{cases}$$

Let $\tilde{U}(s)$ be the Laplace Stieltjes transform of $U(u)$. Note that the Laplace transform of the ruin probability $\Psi(u)$, $\hat{\Psi}(s)$, equals the Laplace Stieltjes transform of the function $U(u)$, $\tilde{U}(s)$,

$$\hat{\Psi}(s) = \mathcal{L}(\Psi(u))(s) = \int_0^\infty e^{-su}\Psi(u)du = \int_0^\infty e^{-su}dU(u) = \tilde{U}(s).$$

The key point of the proof is to show that $\tilde{U}(s)$ behaves asymptotically at zero as $ks^{\rho-2}$. Then, using the Karamata Tauberian Theorem and the Monotone Density Theorem (Bingham et al., 1987), the result follows.

The analysis of the asymptotic behavior of $\hat{\Psi}(s)$ follows the same path as in the classical case study. Recall that Theorem 2.1. (Paulsen and Gjessing, 1997) states that the ruin probability is the solution of the equation $A\Psi(u) = 0$ together with the boundary conditions (see Appendix). The infinitesimal generator of the ruin probability $\Psi(u)$ is given by

$$A\Psi(u) = \frac{\sigma^2}{2}u^2\Psi''(u) + (au+c)\Psi'(u) + \lambda\int_0^\infty (\Psi(u-y) - \Psi(u))\,dF(y).$$

Since $\Psi(u-y) = 1$ for any $u < y$, and $\int_0^\infty dF(y) = 1$, the equation is equivalent to

$$\frac{\sigma^2}{2}u^2\Psi''(u) + (au+c)\Psi'(u) + \lambda\int_0^u \Psi(u-y)\,dF(y) - \lambda\Psi(u) = 0.$$

The Laplace transform of this equation is

$$\frac{\sigma^2}{2}\frac{d^2(s^2\hat{\Psi}(s))}{ds^2} - a\frac{d(s\hat{\Psi}(s))}{ds} + cs\hat{\Psi}(s) - \lambda\hat{\Psi}(s) + \lambda\hat{\Psi}(s)\mathcal{F}(s) + \frac{\lambda}{s}(1-\mathcal{F}(s)) = c\Psi(0),$$

and after differentiation becomes

$$\frac{\sigma^2 s^2}{2}\hat{\Psi}''(s) + (2s\sigma^2 - as)\hat{\Psi}'(s) + (cs - \lambda + \lambda\mathcal{F}(s) + \sigma^2 - a)\hat{\Psi}(s) = c\Psi(0) - \frac{\lambda}{s}(1-\mathcal{F}(s)).$$

Thus, the equation to be analyzed has the form

$$s^2 y'' + p(s) s y' + q(s) y = g(s), \tag{5}$$

with p, q and g holomorphic functions of the form

$$p(s) = p_0 = \frac{2(2\sigma^2 - a)}{\sigma^2}$$

$$q(s) = q_0 + q_1(s) = \frac{2(\sigma^2 - a)}{\sigma^2} + q_1(s)$$

$$g(s) = g_0 + g_1(s) = \frac{2(c\Psi(0) - \lambda\mu)}{\sigma^2} + g_1(s).$$

Due to the fact that $s = 0$ is a regular singular point of the homogeneous equation

$$s^2 y'' + p(s) s y' + q(s) y = 0,$$

the solution has the form

$$\hat{y}(s) = s^\rho \sum_{k=0}^{\infty} c_k s^k = \sum_{k=0}^{\infty} c_k s^{\rho+k}, \tag{6}$$

where the coefficients satisfy the recurrence system of equations $c_0 = 1$ and

$$c_k f(\rho + k) + c_{k-1} f_1(\rho + k - 1) + \cdots + c_0 f_k(\rho) = 0,$$

with

$$f(\rho) = \rho(\rho - 1) + p_0 \rho + q_0,$$

as in Fedoryuk (1991). The first of these equations $c_0 f(\rho) = 0$ is equivalent to

$$\rho^2 + \frac{3\sigma^2 - 2a}{\sigma^2}\rho + \frac{2\sigma^2 - 2a}{\sigma^2} = 0.$$

If $2\sigma^2 \neq a$, the solutions of the homogeneous equation are of the form

$$\hat{y}_1(s) = s^{-1}\gamma_1(s) \quad \hat{y}_2(s) = s^{-2+\rho}\gamma_2(s),$$

where $\gamma_1(0) = \gamma_2(0) = 1$. Using variation of parameters, one can show that the solutions of the non-homogeneous equation (5) have the form

$$\hat{y} = c_1 s^{-1}\gamma_1(s) + c_2 s^{-2+\rho}\gamma_2(s) + c_3\gamma_3(s),$$

under the condition $\rho < 2$, with c_1, c_2, c_3 real constants, γ_1, γ_2 and γ_3 holomorphic functions and $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$. The asymptotic behavior at zero of the solution of equation (5) describes the asymptotic behavior at zero of $\hat{\Psi}(s)$, consequently of $\tilde{U}(s)$. The leading term of this linear combination dictates the asymptotic behavior of the solution as $s \to 0$. Two cases can be distinguished.

If the leading term of the linear combination is s^{-1} then by Karamata Tauberian and Monotone Density theorem

$$\Psi(u) \sim \frac{\alpha\gamma(1/u)}{\Gamma(2)} \quad \text{as} \quad u \to \infty.$$

Hence

$$\lim_{u\to\infty} \Psi(u) = \frac{\alpha}{\Gamma(2)},$$

where α is a real constant. In other words, the ruin probability has a constant asymptotic behavior, as $u \to \infty$. Obviously, in this case, the function does not satisfy the boundary conditions from Paulsen and Gjessing (1997) theorem, so it is not a solution that can be related to the ruin probability.

In the second case, if $s^{-2+\rho}$ is the leading term, then

$$\tilde{U}(s) \sim \alpha s^{-2+\rho}\gamma(s) \quad \text{as} \quad s \to 0.$$

The Karamata-Tauberian Theorem implies

$$U(u) \sim \frac{\alpha u^{2-\rho}\gamma(1/u)}{\Gamma(3-\rho)}, \quad \text{as} \quad u \to \infty.$$

$\Psi(u)$ is monotone, $\alpha \in \mathbf{R}$ and $\rho \in \mathbf{R}$, hence the Monotone Density Theorem implies

$$\Psi(u) \sim \frac{\alpha(2-\rho)u^{2-\rho-1}\gamma(1/u)}{\Gamma(3-\rho)}, \quad \text{as} \quad u \to \infty.$$

Since $\Psi(u)$ must decay, ρ needs to satisfy the condition $2-\rho-1 < 0$. The conclusion is that

$$\Psi(u) = K u^{1-\rho}\gamma(1/u), \quad \text{as} \quad u \to \infty, \quad \text{for} \quad 1 < \rho < 2$$

or

$$\lim_{u\to\infty} \Psi(u)u^{\rho-1} = K, \quad \text{as} \quad u \to \infty, \quad \text{for} \quad 1 < \rho < 2$$

where $K = \frac{\alpha(2-\rho)}{\Gamma(3-\rho)}$. □

Remark 2. *It is conjectured that if $\rho \leq 1$, then $\Psi(u) = 1$ for all u.*

4 Conclusion

This paper illustrates how Laplace transforms and Karamata Tauberian arguments can be used effectively in the analysis of the asymptotic behavior of the ruin probabilities. For example, the classical Cramer Lundberg result can be obtained using elementary properties of the Laplace transform. When uncertain returns on investments are modeled by a geometric Brownian Motion, the asymptotic behavior of the ruin probability can be derived using this methodology. Given the analytic nature of these tools, the results obtained in this paper can be generalized to include different claim inter-arrival times or investment strategies. The results presented in this paper are part of the PhD thesis of Corina Constantinescu at Oregon State University.

5 Appendix

This is a short summary of some of the technical results used in the paper.

Theorem 3 (Paulsen and Gjessing (1997)). *If $\Psi(u)$ is a bounded and twice continuous differentiable function defined for $u \geq 0$ that solves $\mathbf{A}\Psi(u) = 0$ on $u > 0$ together with the boundary conditions:*

$$\Psi(u) = 1, \quad for\ u < 0$$

$$\lim_{u \to \infty} \Psi(u) = 0$$

then the solution is

$$\Psi(u) = \boldsymbol{P}(T_u < \infty).$$

Definition 1. *Let l be a positive measurable function, defined in some neighborhood $[M, \infty)$ of infinity, and satisfying*

$$l(\lambda x)/l(x) \to 1, \quad as \quad x \to \infty, \quad \forall \lambda > 0,$$

then l is said to be slowly varying in Karamata's sense (Bingham et al., 1987).

Theorem 4 (Karamata Tauberian Theorem). *Let U be a non-decreasing right-continuous function on $\boldsymbol{R}$ with $U(x) = 0$ for all $x < 0$. If l varies slowly and $c \geq 0, \rho \geq 0$ the following are equivalent:*

$$U(x) \sim cx^{\rho} l(x)/\Gamma(1+\rho), \quad (x \to \infty),$$

$$\tilde{U}(s) \sim cs^{-\rho} l(1/s), \quad (s \to 0_+),$$

where $\tilde{U}$ denotes the Laplace-Stieltjes transform (Bingham et al., 1987).

Definition 2. *A function f is ultimately monotone if there exists y such that for any $x > y$, $f(x)$ is monotone.*

Theorem 5 (Monotone Density Theorem). *Let $U(x) = \int_0^x u(y)dy$. If*

$$U(x) \sim cx^\rho l(x), \quad x \to \infty,$$

where $c \in \mathbf{R}$, $l \in \mathbf{R}_0$, and if u is ultimately monotone, then

$$u(x) \sim c\rho x^{\rho-1} l(x), \quad x \to \infty,$$

(Bingham et al., 1987).

References

N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular variation.* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1987.

H. Cramer. On the mathematical theory of risk. *Skandia Jubilee Volume, Stockholm*, 1930.

P. Embrechts, C. Kluppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin, 1997.

M. Fedoryuk. *Asymptotic Analysis.* Springer-Verlag, 1991.

A. Frolova, Y. Kabanov, and S. Pergamenshchikov. In the insurance business risky investments are dangerous. *Finance and stochastics*, 6:227–235, 2002.

H.. Gerber. Martingales in risk theory. *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 73:205–216, 1973.

J. Paulsen and H. Gjessing. Ruin theory with stochastic return on investments. *Advanced Applied Probability*, 29:965–985, 1997.

T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels. *Stochastic Processes for Insurance and Finance.* Wiley Series in Probability and Statistics, New York, 1999.

S. Ross. *Stochastic Processes.* Wiley, New York, 1996.

H. Schmidli. Cramer-lundberg approximations for ruin probabilities of risk processes perturbed by diffusion. *Insurance: Mathematics and Econos*, 16: 135–149, 1995.