Capital Allocation in Insurance: Economic Capital and the Allocation of the Default Option Value*

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Introduction

- Many different methods (risk measures) for measuring solvency risk - VaR, 1 year ruin probability, infinite horizon ruin probability, TailVaR, Expected Policyholder Deficit, Default Option value (Insolvency Exchange Option value)

- Many different approaches to allocating capital to line of business - proportional to selected risk measure, proportional to liabilities, marginal allocations (Merton and Perold and Myers and Read), equal expected returns to capital, covariance of losses, capital allocation irrelevant

Aims of Paper

- To consider economic capital for a multi-line insurer and the fair (arbitrage-free) allocation to lines of business

- For a model with assets, liabilities and insolvency exchange option fairly priced (arbitrage-free), complete markets and no market frictions

  - Derive closed form expressions for by-line and insurer insolvency exchange option (default option)

  - Illustrate results with numerical examples

Economic Balance Sheet

Balance Sheet	Initial Value	End of Period Payoff
Assets	V	$V(T)$
Liabilities	$L-D$	$L(T)-(L(T)-V(T))^{+}$
Equity	$S+D$	$\left[V(T)-L(T)+(L(T)-V(T))^{+}\right]$

Table 1: *Total Insurer Economic Balance Sheet*

- V is (time 0) value of assets, L is value of liabilities assuming no default, D is option value of loss arising from insolvency

Economic Capital

- Surplus (market value of assets minus value of liabilities ignoring default) plus the insolvency exchange option (default option value)

$$V(0) - L(0) + D(0)$$

- Risk measure is the insolvency exchange option value

$$\begin{aligned} D(0) &= E^Q\left[e^{-rT}\left[L(T) - V(T)\right]^+ |\mathcal{F}_0\right] \\ &= E^Q\left[e^{-rT}\left[L(T) - V(T)\right] | \frac{V(T)}{L(T)} < 1\right] \Pr^Q\left[\frac{V(T)}{L(T)} < 1\right] \end{aligned}$$

Insolvency Exchange Option Value

- Reflects both severity and probability

- Q probabilities ("risk neutral") and not P probabilities

- Measures the asset-liability mismatch (both assets and liabilities)

- Underestimated(?) based on current insurer asset allocations and liability uncertainty

Capital Allocation - Insurer Default Option Value

- Assumptions

$$dL_i(t) = \mu_i L_i(t)\,dt + \sigma_i L_i(t)\,dB^i(t) \quad \text{for } i = 1, \ldots, M$$

$$\Lambda(t) \triangleq \frac{V(t)}{L(t)}$$

$$d\Lambda(t) = \mu_\Lambda \Lambda(t)\,dt + \sigma_\Lambda \Lambda(t)\,dB^\Lambda(t)$$

- Allows derivation of closed form expression for default option value by line of business

Capital Allocation - Insurer Default Option Value

- End-of-period payoff to line of business i is well defined with equal priority

$$\frac{L_i(T)}{L(T)}V(T) \quad \text{if } L(T) > V(T) \ \left(\text{or } \frac{V(T)}{L(T)} \leq 1\right)$$
$$L_i(T) \quad \text{if } L(T) \leq V(T) \ \left(\text{or } \frac{V(T)}{L(T)} > 1\right)$$

$$\begin{aligned} D_i(t) &= E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\frac{V(T)}{L(T)}\right]^+ |\mathcal{F}_t\right] \\ &= E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\Lambda(T)\right]^+ |\mathcal{F}_t\right] \end{aligned}$$

Capital Allocation - Insurer Default Option Value "Adds Up"

$$
\begin{aligned}
\sum_{i=1}^{M} D_i(t) &= \sum_{i=1}^{M} E^Q\left[e^{-r(T-t)} L_i(T)\left[1-\Lambda(T)\right]^+ |\mathcal{F}_t\right] \\
&= \sum_{i=1}^{M} E^Q\left[e^{-r(T-t)} L_i(T)\left[1-\frac{V(T)}{L(T)}\right]^+ |\mathcal{F}_t\right] \\
&= E^Q\left[e^{-r(T-t)}\left[L(T)-V(T)\right]^+ |\mathcal{F}_t\right] \\
&= D(t)
\end{aligned}
$$

Insurer Default Option Value By Line - Closed Form Expression

- Change of measure (Radon-Nikodym derivative)

$$\frac{dQ^i}{dQ}|\mathcal{F}_T = Z_i\left(t\right) = \frac{1}{L_i\left(0\right)}\frac{L_i\left(t\right)e^{\left(r-\mu_i\right)t}}{e^{rt}}$$

- Value becomes

$$D_i\left(0\right) = L_i\left(0\right)e^{\mu_i T}E^{Q_i}\left[e^{-rT}\left(1-\Lambda\left(T\right)\right)^+|\mathcal{F}_0\right]$$

Insurer Default Option Value By Line - Closed Form Expression

- Exchange option closed form (used to compute by-line allocation using change of measure)

$$\begin{aligned} M(0) &= E^{Q}\left[e^{-rT}\left[1-\Lambda(T)\right]^{+}|\mathcal{F}_0\right] \\ &= e^{-rT}N(-d_{20})-\Lambda(0)\,e^{-(r-\mu_\Lambda)T}N(-d_{10}) \end{aligned}$$

$$d_{10}=\frac{\ln\Lambda(0)+\left(\mu_\Lambda+\frac{1}{2}\sigma_\Lambda^2\right)T}{\sigma_\Lambda\sqrt{T}}$$

$$d_{20}=d_{10}-\sigma_\Lambda\sqrt{T}$$

Insurer Default Option Value By Line - Closed Form Expression

$$\begin{aligned} D_i(0) &= L_i(0)\, e^{\mu_i T} E^{Q_i}\left[e^{-rT}(1-\Lambda(T))^{+} \,|\mathcal{F}_0\right] \\ &= L_i(0)\, e^{\mu_i T} M^i(0) \\ D(0) &= \sum_{i=1}^{M} D_i(0) \end{aligned}$$

$M^i(0)$ evaluate with $M(0)$ formula - μ_Λ replaced by $\mu_\Lambda^i = \mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda$
$d\Lambda(t) = \mu_\Lambda \Lambda(t)\, dt + \sigma_\Lambda \Lambda(t)\left(d\widetilde{B}^\Lambda(t) + \rho_{i\Lambda}\sigma_i dt\right)$
$\widetilde{B}^\Lambda(t)$ Brownian motion under Q_i and $dB^i(t)\, dB^\Lambda(t) = \rho_{i\Lambda} dt$

Comments on Capital Allocation - Myers and Read (2001) with constant d_i

- Myers and Read (2001) results should be interpreted as sensitivities to maintain the default option value per unit of liabilities and not capital allocations ($d_i = \frac{\partial D}{\partial L_i}$)

- Note $\frac{\partial}{\partial L_i}\left(\frac{D}{L}\right) = \frac{1}{L}\left[\frac{\partial D}{\partial L_i} - \frac{D}{L}\right]$ and selecting s_i so that $\frac{\partial D}{\partial L_i} = \frac{D}{L}$ for all i ensures $\frac{\partial}{\partial L_i}\left(\frac{D}{L}\right) = 0$. But note this is for a static balance sheet.

- Constant d_i required to maintain balance sheet $\frac{D}{L}$ and s_i gives incremental capital required from shareholders for small changes in each line (but these are not the actual allocations for current balance sheet)

Internal By Line of Business Balance Sheet

Balance Sheet	Initial Value	End of Period Payoff
Assets	V_i	$V_i(T)$
Liabilities	$L_i - D_i$	$L_i(T) - L_i(T)\left(1 - \frac{V(T)}{L(T)}\right)^+$
Equity	$S_i + D_i$	$\left[V_i(T) - L_i(T) + L_i(T)\left(1 - \frac{V(T)}{L(T)}\right)^+\right]$

Table 2: *Internal Balance Sheet for Line of Business* i

Comments on Internal By Line of Business Balance Sheet

- Assumes equal priority to assets in the event of insolvency for outstanding liabilities by line of business

- Allocation of surplus is arbitrary (capital allocation irrelevance) - only an internal allocation (many different allocations will "add up".

- Allocation of insolvency exchange option value (default option value) reflects the by-line loss in the event of insolvency and has economic significance (pricing, fair valuation)

Example from Myers and Read (2001)

		Ratio to Liabilities	Standard Deviation	Correlations		
				Line 1	Line 2	Line 3
Line 1	$100	33%	10%	1.00	0.50	0.50
Line 2	$100	33%	15%	0.50	1.00	0.50
Line 3	$100	33%	20%	0.50	0.50	1.00
Liabilities	$300	100%	12.36%	0.74	0.81	0.88
Assets	$450	150%	15%	-0.20	-0.20	-0.20
Surplus	$150	50%				

Table 3: *Data from Table 2 of Myers and Read*

Example from Myers and Read (2001)

		Covariance with Liabilities	Covariance with Assets	μ_{Λ}^{i}
Line 1	$100	0.0092	-0.0030	0.0076
Line 2	$100	0.0150	-0.0045	0.0003
Line 3	$100	0.0217	-0.0060	-0.0079
Liabilities	$300	0.0153	-0.0045	0.0000
Assets	$450		0.0225	
Surplus	$150		σ_{Λ}	0.2163

Table 4: *Parameters for Table 2 Data of Myers and Read*

Example from Myers and Read (2001)

	Partial Derivatives Uniform Default Value Myers-Read/Sherris-van der Hoek Comparison			
	MR d_i %	MR s_i%	SvDH d_i%	SvDH s_i%
Line 1	0.3112	37.75	0.3119	37.53
Line 2	0.3112	49.55	0.3119	49.50
Line 3	0.3112	62.90	0.3119	62.98
Total	0.3112	50	0.3119	50.00

Table 5: *Line by Line Sensitivities Table 2 Data of Myers and Read - Lognormal assumption*

Note: s_i are sensitivities (not allocations) to maintain value of d with $d_i = d$

Example from Myers and Read (2001)

	Capital Allocations Uniform Default Value Myers-Read/Sherris-van der Hoek Comparison			
	MR d_i %	MR s_i%	SvDH $\widetilde{d}_i$%	SvDH $\widetilde{s}_i$%
Line 1	0.3112	37.75	0.2852	not unique
Line 2	0.3112	49.55	0.3102	not unique
Line 3	0.3112	62.90	0.3404	not unique
Total	0.3112	50	0.3119	50

Table 6: *Line by Line Allocations Table 2 Data of Myers and Read - Lognormal assumption*

Data from Panjer (2001)

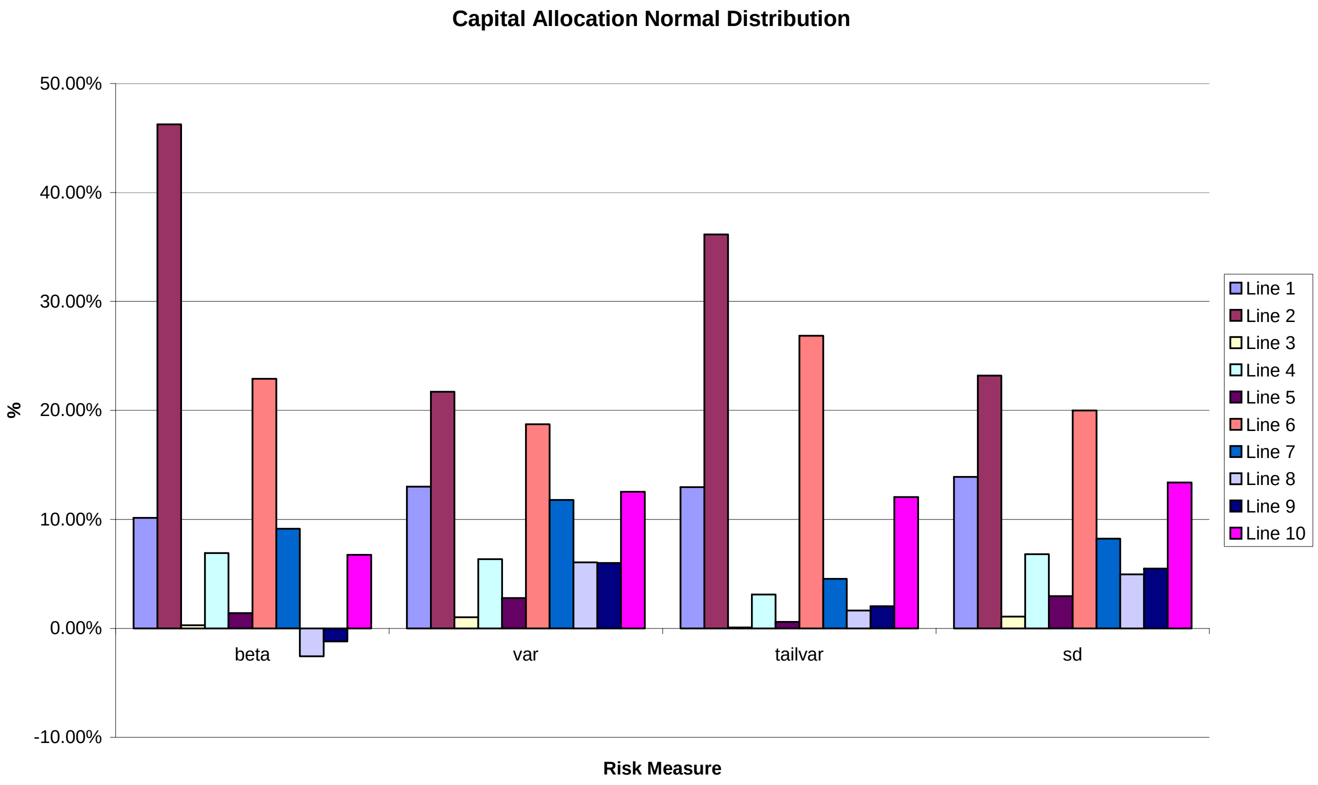
Line	Amount	PerCent	Standard Deviation (Dollar)	Standard Deviation (Percent)
Line 1	36.00	9.6	2.69	7.47
Line 2	120.40	32.3	4.49	3.73
Line 3	1.30	0.3	0.21	16.12
Line 4	52.42	14.0	1.32	2.51
Line 5	0.70	0.2	0.57	82.14
Line 6	48.09	12.9	3.87	8.05
Line 7	47.40	12.7	1.59	3.36
Line 8	8.08	2.2	0.96	11.85
Line 9	8.64	2.3	1.06	12.29
Line 10	50.15	13.4	2.59	5.17
Liabilities	373.18	100	6.73	1.80
Assets	400.42	107	60.06	15.00
Surplus	27.24	7		

Table 7: *Liabilities and Standard Deviations*

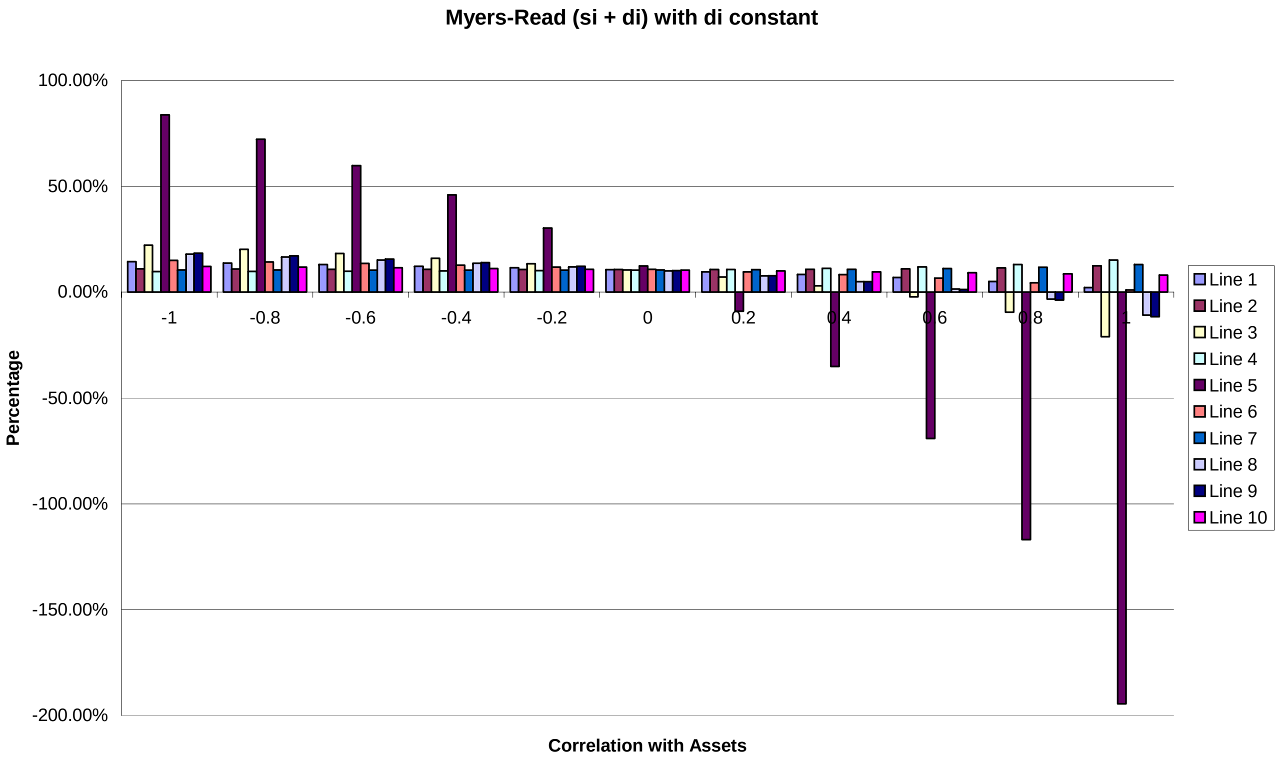
Data from Panjer (2001)

Line 1	Line 2	Line 3	Line 4	Line 5	Line 6	Line 7	Line 8	Line 9	Line 10
1.00	0.00	0.12	-0.02	0.18	-0.26	-0.12	0.11	0.08	-0.03
0.00	1.00	0.05	0.27	0.02	0.08	0.16	-0.21	-0.17	-0.15
0.12	0.05	1.00	0.01	-0.11	0.10	0.03	-0.12	-0.09	-0.12
-0.02	0.27	0.01	1.00	0.22	0.05	0.09	-0.11	0.13	-0.23
0.18	0.02	-0.11	0.22	1.00	-0.11	0.01	-0.03	0.14	-0.01
-0.26	0.08	0.10	0.05	-0.11	1.00	0.07	-0.09	-0.46	-0.16
-0.12	0.16	0.03	0.09	0.01	0.07	1.00	-0.25	0.08	0.14
0.11	-0.21	-0.12	-0.11	-0.03	-0.09	-0.25	1.00	-0.16	-0.16
0.08	-0.17	-0.09	0.13	0.14	-0.46	0.08	-0.16	1.00	0.21
-0.03	-0.15	-0.12	-0.23	-0.01	-0.16	0.14	-0.16	0.21	1.00
0.25	0.69	0.09	0.35	0.16	0.40	0.39	-0.18	-0.08	0.18

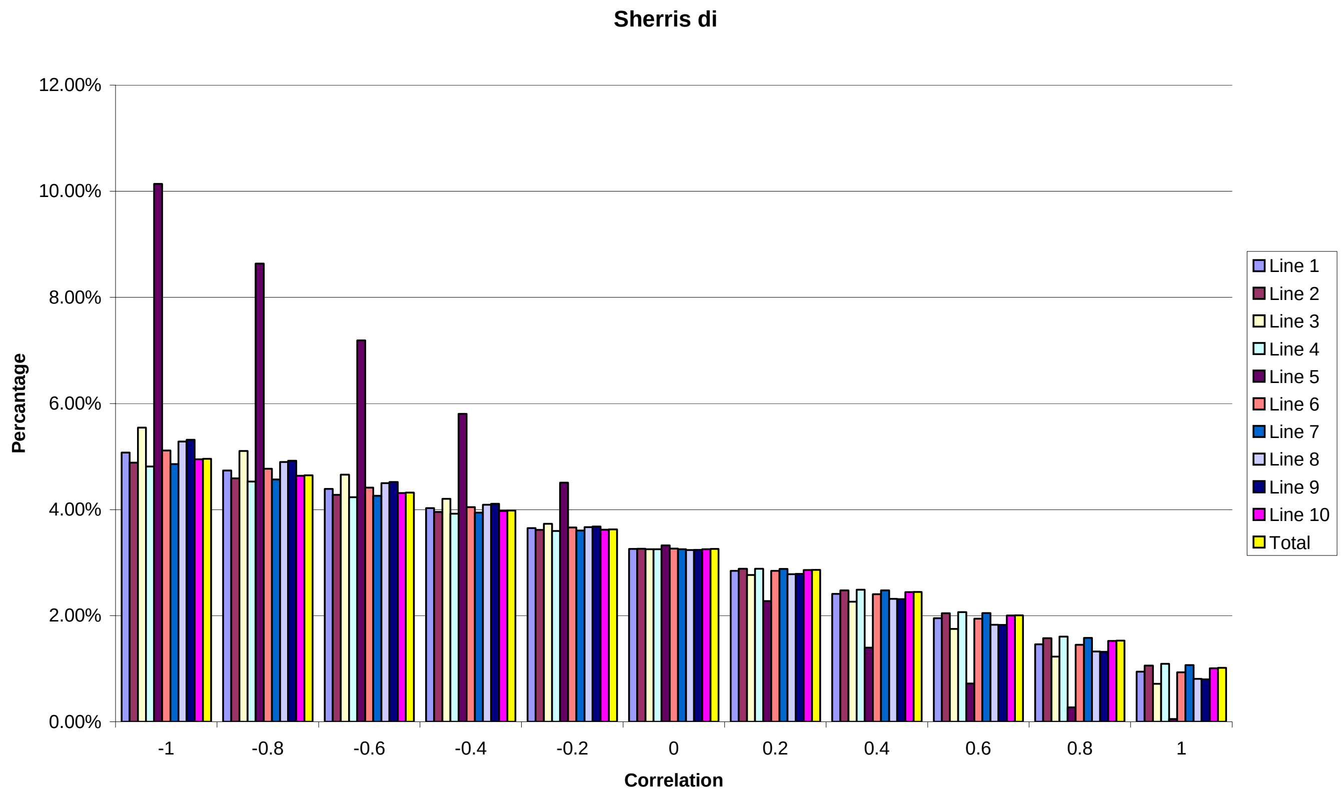
Table 8: *Correlations by line of business for Panjer Data*



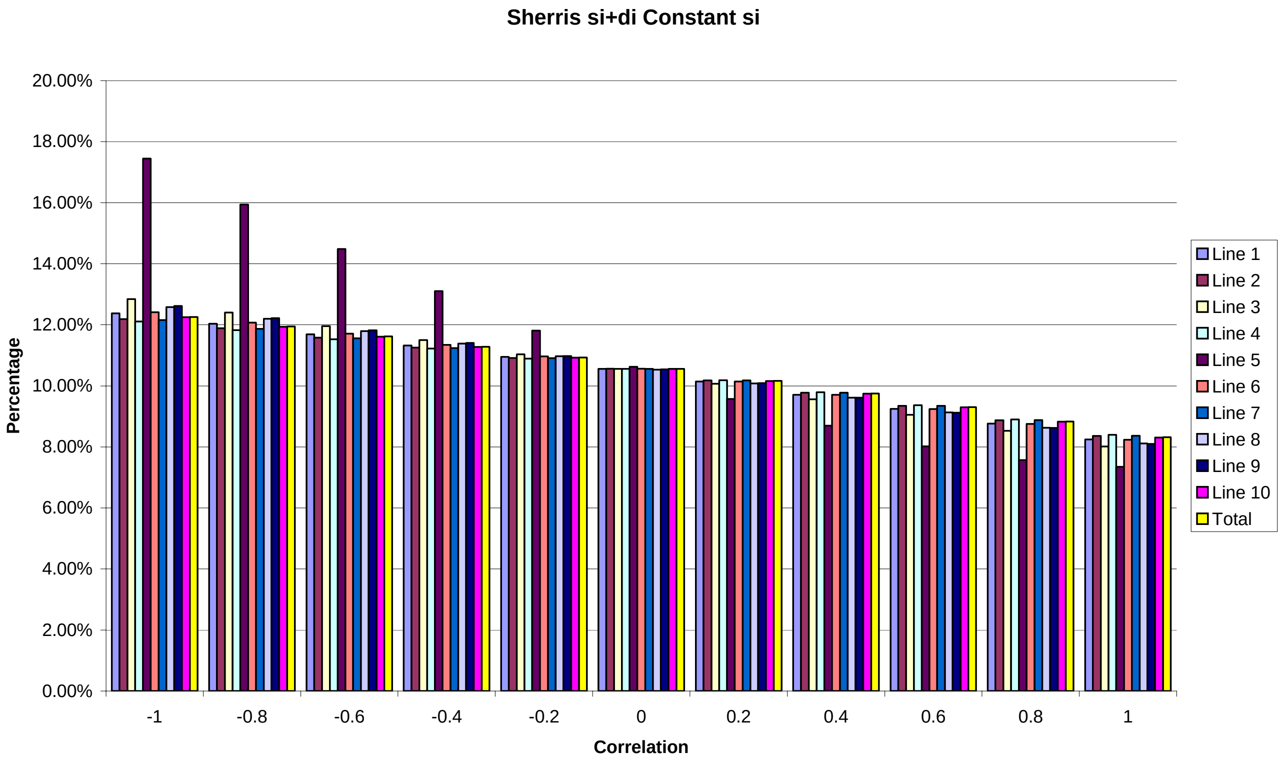
Capital Allocation with Panjer data and different Risk Measures



Myers Read Allocations (Panjer data - Log-normal assumption)



Sherris-van der Hoek Default Option Values (Allocations not Sensitivities)



Sherris-van der Hoek Allocations with Constant Solvency Ratio By Line

Conclusions

- We have derive closed form expression for the by-line and total insolvency exchange option value assuming arbitrage-free and complete markets

- Capital allocations not unique but insolvency exchange option value reflects the insurer balance sheet risk and are used for (arbitrage-free) by-line pricing

- Myers and Read (2001) results are (static) sensitivities and not capital allocations (capital required to maintain ratio of insurer default option value to liabilities for small changes to a single line of business)