

# ASTAM Formula Sheet

Version: Jan 26, 2025

$P(z)$  denotes the probability generating function.  $M(z)$  denotes the moment generating function.

$Q_\alpha$  denotes the  $\alpha$ -quantile of a distribution, also known as the  $\alpha$ -Value at Risk

$ES_\alpha[X]$  denotes the  $\alpha$ -Expected Shortfall of  $X$ . This is also known as the  $\alpha$ -TailVaR, or the  $\alpha$ -CTE.

The distribution function of the standard normal distribution is denoted  $\Phi(x)$ . The probability density function of the standard normal distribution is denoted  $\phi(x)$ .

The  $q$ -quantile of the standard normal distribution is denoted  $z_q$ , that is  $\Phi(z_q) = q$ .

For counting distributions,  $p_k$  denotes the probability function.

## Continuous distributions

### Pareto( $\alpha, \theta$ ) Distribution

$$f(x) = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}, \quad F(x) = 1 - \left(\frac{\theta}{\theta+x}\right)^\alpha,$$

$$E[X] = \frac{\theta}{\alpha-1}, \quad \alpha > 1, \quad \text{Var}[X] = \left(\frac{\theta}{\alpha-1}\right)^2 \frac{\alpha}{\alpha-2}, \quad \alpha > 2,$$

$$E[X \wedge x] = \frac{\theta}{\alpha-1} \left(1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right), \quad \alpha > 1,$$

$$E[X^k] = \frac{k! \theta^k}{(\alpha-1)(\alpha-2)\dots(\alpha-k)}, \quad k = 1, 2, \dots, \quad \alpha > k,$$

$$E[X - Q | X > Q] = \frac{\theta + Q}{\alpha - 1}, \quad \alpha > 1.$$

### Lognormal( $\mu, \sigma$ ) Distribution

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \quad F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right),$$

$$E[X] = e^{\mu+\sigma^2/2}, \quad \text{Var}[X] = e^{2\mu+2\sigma^2}(e^{\sigma^2} - 1),$$

$$E[X^k] = e^{k\mu+k^2\sigma^2/2},$$

$$E[(X \wedge x)^k] = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right) \Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k \left(1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right),$$

$$ES_\alpha[X] = \frac{e^{\mu+\sigma^2/2}}{1-\alpha} \Phi(z_{1-\alpha} + \sigma).$$

### Exponential( $\theta$ ) Distribution

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \quad F(x) = 1 - e^{-x/\theta},$$

$$E[X] = \theta, \quad \text{Var}[X] = \theta^2,$$

$$E[X^k] = k! \theta^k, \quad k = 1, 2, \dots,$$

$$E[X \wedge x] = \theta(1 - e^{-x/\theta}),$$

$$E[X - Q | X > Q] = \theta,$$

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta}.$$

### Gamma( $\alpha, \theta$ ) Distribution

$$f(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)},$$

$$E[X] = \alpha\theta, \quad \text{Var}[X] = \alpha\theta^2,$$

$$E[X^k] = \alpha(\alpha+1)\dots(\alpha+k-1)\theta^k, \quad k = 1, 2, \dots,$$

$$M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}.$$

### Chi-squared( $\nu$ ) Distribution

Gamma distribution with  $\alpha = \nu/2$  and  $\theta = 2$ .

$\nu \in \mathbb{N}^+$  is the degrees of freedom parameter.

$$E[X] = \nu, \quad \text{Var}[X] = 2\nu$$

$$M_X(t) = (1 - 2t)^{-\nu/2}, \quad t < \frac{1}{2}.$$

### Beta( $a, b$ ) Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,$$

$$E[X] = \frac{a}{a+b}, \quad \text{Var}[X] = \frac{ab}{(a+b)^2(a+b+1)}.$$

### Normal( $\mu, \sigma^2$ ) Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$E[X] = \mu, \quad \text{Var}[X] = \sigma^2,$$

$$ES_\alpha[X] = \mu + \frac{\sigma}{1-\alpha} \phi(z_\alpha),$$

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}.$$

### Weibull( $\theta, \tau$ ) Distribution

$$f(x) = \frac{\tau x^{\tau-1}}{\theta^\tau} e^{-(x/\theta)^\tau}, \quad F(x) = 1 - e^{-(x/\theta)^\tau},$$

$$E[X] = \theta \Gamma\left(1 + \frac{1}{\tau}\right), \quad \text{Var}[X] = \theta^2 \left( \Gamma\left(1 + \frac{2}{\tau}\right) - \left( \Gamma\left(1 + \frac{1}{\tau}\right) \right)^2 \right).$$

## Counting Distributions

### Poisson( $\lambda$ ) Distribution

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!}, \quad a = 0, \quad b = \lambda,$$

$$E[N] = \lambda, \quad \text{Var}[N] = \lambda,$$

$$P_N(z) = \exp\{\lambda(z-1)\}, \quad M_N(z) = \exp\{\lambda(e^z - 1)\}.$$

### Binomial( $m, q$ ) Distribution

$$p_k = \binom{m}{k} q^k (1-q)^{m-k}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},$$

$$E[N] = mq, \quad \text{Var}[N] = mq(1-q)$$

$$P_N(z) = (1 + q(z-1))^m, \quad M_N(z) = (1 + q(e^z - 1))^m.$$

### Bernoulli( $q$ ) Distribution

Binomial Distribution with  $m = 1$ .

### Negative Binomial( $r, \beta$ ) Distribution

$$p_0 = \left(\frac{1}{1+\beta}\right)^r, \quad p_k = \frac{r(r+1)\cdots(r+k-1)}{k!} \left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r, \quad k = 1, 2, \dots,$$

$$a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},$$

$$E[N] = r\beta, \quad \text{Var}[N] = r\beta(1+\beta),$$

$$P_N(z) = (1 - \beta(z-1))^{-r}, \quad M_N(z) = (1 - \beta(e^z - 1))^{-r}.$$

### Geometric Distribution

Negative Binomial Distribution with  $r = 1$ ;

## Critical Values for Kolmogorov-Smirnov Test

$\alpha$	Critical Value
0.1	$1.22/\sqrt{n}$
0.05	$1.36/\sqrt{n}$
0.01	$1.63/\sqrt{n}$

## Recursions for Compound Distributions

$$\text{For } N \sim (a, b, 0): \quad f_S(x) = \frac{\sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)}$$

$$\text{For } N \sim (a, b, 1): \quad f_S(x) = \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)}$$

## Empirical Bayes Credibility

Empirical Bayes parameter estimation for the Bühlmann model:

$$\bar{X}_i = \frac{\sum_{j=1}^n X_{ij}}{n}, \quad \bar{X} = \frac{\sum_{i=1}^r \bar{X}_i}{r},$$

$$\hat{v} = \frac{1}{r(n-1)} \sum_{i=1}^r \sum_{j=1}^n \left( X_{ij} - \bar{X}_i \right)^2, \quad \hat{a} = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i - \bar{X})^2 - \frac{\hat{v}}{n},$$

$$\hat{\mu} = \bar{X}, \quad \hat{Z}_i = \frac{n}{n + \hat{v}/\hat{a}}.$$

Empirical Bayes parameter estimation for the Bühlmann-Straub model:

$$m_i = \sum_{j=1}^{n_i} m_{ij}, \quad m = \sum_{i=1}^r m_i, \quad \bar{X}_i = \frac{\sum_{j=1}^{n_i} m_{ij} X_{ij}}{m_i}, \quad \bar{X} = \frac{\sum_{i=1}^r m_i \bar{X}_i}{m},$$

$$\hat{v} = \frac{\sum_{i=1}^r \sum_{j=1}^{n_i} m_{ij} \left( X_{ij} - \bar{X}_i \right)^2}{\sum_{i=1}^r (n_i - 1)}, \quad \hat{a} = \frac{\sum_{i=1}^r m_i (\bar{X}_i - \bar{X})^2 - \hat{v}(r-1)}{m - \frac{1}{m} \sum_{i=1}^r m_i^2}$$

$$\hat{Z}_i = \frac{m_i}{m_i + \hat{v}/\hat{a}}, \quad \hat{\mu} = \frac{\sum_{i=1}^r \hat{Z}_i \bar{X}_i}{\sum_{i=1}^r \hat{Z}_i}.$$

# Extreme Value Theory

## The Gumbel Distribution

$$F(x) = \exp \left\{ - \exp \left( - \frac{x - \mu}{\theta} \right) \right\}, \quad \theta > 0.$$

## The Fréchet Distribution

$$F(x) = \exp \left\{ - \left( \frac{x - \mu}{\theta} \right)^{-\alpha} \right\}, \quad x > \mu; \alpha > 0; \theta > 0.$$

## The Weibull EV Distribution

$$F(x) = \exp \left\{ - \left( \frac{\mu - x}{\theta} \right)^\tau \right\}, \quad x < \mu; \tau > 0; \theta > 0.$$

## The Generalized Extreme Value Distribution

The distribution function is  $H(x)$  where

$$H_\xi(x) = \begin{cases} \exp \left( -(1 + \xi x)^{-\frac{1}{\xi}} \right) & \xi \neq 0, \xi x > -1, \\ \exp(-e^{-x}) & \xi = 0. \end{cases}$$

The GEV can be adjusted for scale and location, to give  $H_{\xi, \mu, \theta}$  where

$$H_{\xi, \mu, \theta}(x) = \begin{cases} \exp \left( -(1 + \xi(x - \mu)/\theta)^{-\frac{1}{\xi}} \right) & \xi \neq 0, (1 + \xi(x - \mu)/\theta) > 0, \\ \exp(-e^{-(x - \mu)/\theta}) & \xi = 0, \end{cases}$$

## The Generalized Pareto Distribution (GPD)

$$G(x) = \begin{cases} 1 - (1 + \xi x/\beta)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1 - e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$$

$$E[X] = \frac{\beta}{1 - \xi} \quad \text{for } 0 < \xi < 1$$

If  $X - d | X > d \sim \text{GPD}(\xi, \beta)$  then

$$Q_\alpha = d + \frac{\beta}{\xi} \left( \left( \frac{S_X(d)}{1 - \alpha} \right)^\xi - 1 \right)$$

$$\text{ES}_\alpha = \frac{1}{1 - \xi} \left( Q_\alpha + \beta - \xi d \right)$$

## The Hill Estimator

$$\hat{\alpha}_j^H = \left( \sum_{k=j}^n \frac{\log(x_{(k)})}{n-j+1} - \log(x_{(j)}) \right)^{-1}$$

(Note: the version of the Hill estimator in QERM is incorrect.)

# Outstanding Claims Reserves

## Functions of development factors

If all claims are settled by the end of DY  $J$ , then

$$\text{for } j = 0, 1, \dots, J-1, \lambda_j = \prod_{k=j}^{J-1} f_k; \quad \lambda_J = 1$$

$$\text{for } j = 0, 1, \dots, J, \quad \beta_j = \frac{1}{\lambda_j}; \quad \gamma_j = \beta_j - \beta_{j-1}$$

## Tests for correlated development factors

$$T_j = r_j \sqrt{\frac{\nu_j}{1 - r_j^2}}, \quad T = \frac{\sum_{\nu_j \geq 3} T_j (\nu_j - 2) / \nu_j}{\sum_{\nu_j \geq 3} (\nu_j - 2) / \nu_j}$$

Under the null hypothesis,  $T \approx N(0, v)$  where  $v = \frac{1}{\sum_{\nu_j \geq 3} ((\nu_j - 2) / \nu_j)}$ .

## Test for calendar year effects

$$Z_k = \min(S_k, L_k); \quad Z = \sum_{k=1}^{I-1} Z_k.$$

Under the null hypothesis, approximately:

$$E[Z_k] = \frac{n_k}{2} - \binom{n_k - 1}{m_k} \frac{n_k}{2^{n_k}}$$

$$\text{Var}[Z_k] = \frac{n_k(n_k - 1)}{4} - \binom{n_k - 1}{m_k} \frac{n_k(n_k - 1)}{2^{n_k}} + E[Z_k] - E[Z_k]^2$$

$$E[Z] = \sum_{k=1}^{I-1} E[Z_k]; \quad \text{Var}[Z] = \sum_{k=1}^{I-1} \text{Var}[Z_k]; \quad \frac{Z - E[Z]}{\sqrt{\text{Var}[Z]}} \sim N(0, 1)$$



## The Bühlmann-Straub Model of Outstanding Claims

$$m_i = \hat{\beta}_{I-i}; \quad s_i^2 = \frac{1}{I-i} \sum_{j=0}^{I-i} \hat{\gamma}_j \left( \frac{X_{i,j}}{\hat{\gamma}_j} - \hat{C}_{i,J} \right)^2$$

$$m = \sum_{i=0}^I m_i; \quad \bar{C} = \frac{\sum_{i=0}^I C_{i,I-i}}{m}$$

$$\hat{v} = \frac{1}{I} \sum_{i=0}^{I-1} s_i^2; \quad \hat{a} = \frac{\sum_{i=0}^I m_i \left( \hat{C}_{i,J} - \bar{C} \right)^2 - I\hat{v}}{m - \frac{1}{m} \sum_{i=0}^I m_i^2}$$

$$Z_i = \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \hat{v}/\hat{a}} \quad \hat{\mu} = \frac{\sum_{i=0}^I Z_i \hat{C}_{i,J}}{\sum_{i=0}^I Z_i}$$

## Mack's Model

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j} \sum_{i=0}^{I-1-j} C_{i,j} \left( f_{i,j} - \hat{f}_j \right)^2 \quad \text{for } j \leq I-2$$

$$\text{Var}[C_{i,J} - C_{i,I-i} | C_{i,I-i}] \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \hat{C}_{i,j}}$$

$$\left( \hat{C}_{i,J} - \text{E}[C_{i,J} | \mathcal{D}_I] \right)^2 \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 S_j}$$

$$\text{MSEP} \left( \hat{R}_I | \mathcal{D}_I \right) \approx \sum_{i=1}^I \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \left( \frac{1}{\hat{C}_{i,j}} + \frac{1}{S_j} \right) + 2 \sum_{i=1}^{I-1} \hat{C}_{i,J} \left( \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 S_j} \right) \left( \sum_{l=i+1}^I \hat{C}_{l,J} \right)$$